Analysis and Control of Multi-Agent Systems

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1 Irreducible Matrices

A matrix $M \in \mathbb{R}^{n \times n}$ is said to be *irreducible* if there does not exist a permutation matrix P and an integer r such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

with $B \in \mathbb{R}^{r \times r}$, $C \in \mathbb{R}^{r \times n-r}$, and $D \in \mathbb{R}^{n-r \times n-r}$.

A *graph of a matrix* $M \in \mathbb{R}^{n \times n}$, denoted $\mathcal{G}(M) = (\mathcal{V}(M), \mathcal{E}(M))$, is a directed graph on n nodes such that there is a directed edge from node i to node j (i.e., $(v_i, v_j) \in \mathcal{E}(M)$) if and only if $[M]_{ij} \neq 0$.

We now give a proof of the theorem stated in class; **a matrix $M \in \mathbb{R}^{n \times n}$ is irreducible if and only if $\mathcal{G}(M)$ is strongly connected.**

M is irreducible $\Rightarrow$ $\mathcal{G}(M)$ is strongly connected

First, assume that $\mathcal{G}(M) = (\mathcal{V}(M), \mathcal{E}(M))$ is *not* strongly connected. This implies that there must exist at least one pair of nodes $u, v \in \mathcal{V}(M)$ such that there *does not exist a directed path from v to u*. This observation will allow us to define 3 special subsets of the node-set $\mathcal{V}(M)$, defined as follows. Let,

$$\begin{aligned} W(u) &= \{s \in \mathcal{V}(M) \,|\, \text{there exists a directed path from } s \text{ to } u\} \cup \{u\}, \\ R(v) &= \{s \in \mathcal{V}(M) \,|\, \text{there exists a directed path from } v \text{ to } s\} \cup \{v\}, \\ Q(u,v) &= \mathcal{V}(M) \setminus (W(u) \cup R(v)). \end{aligned}$$

To visualize these sets, consider the directed graph in Figure 1. The sets $W(u)$, $R(v)$, and $Q(u,v)$ are marked with red, blue, and green nodes respectively. From this picture and the definition of the sets we can arrive at some immediate conclusions:

1. $W(u) \neq \emptyset$; it must contain at least the vertex u.
2. $R(v) \neq \emptyset$; it must contain at least the vertex v.
3. $W(u) \cup R(v) \cup Q(u,v) = \mathcal{V}(M)$.

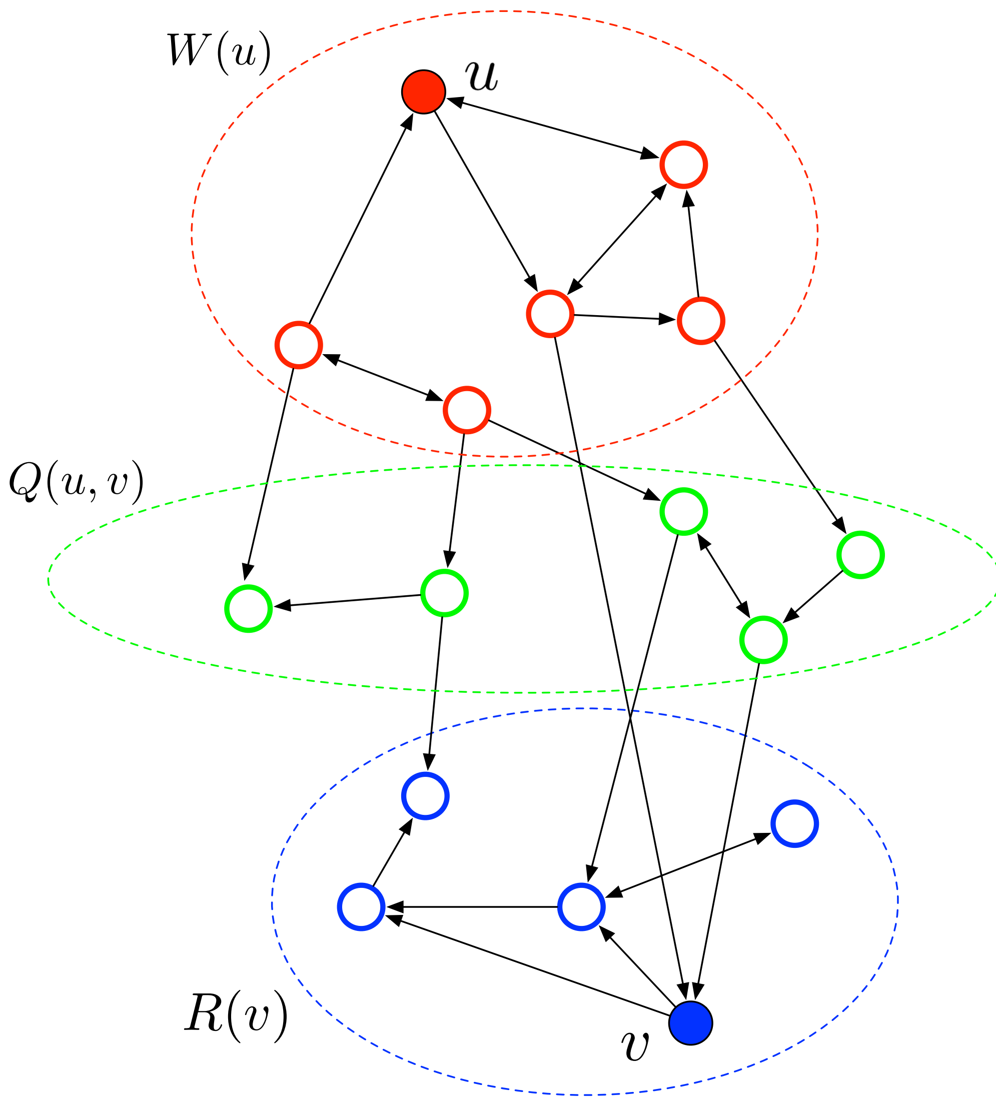

Figure 1: A sketch of the graph used for the proof.

4. $W(u) \cap R(v) = \emptyset$. To show this, assume that $W(u) \cap R(v) = \{q\}$. This means that there is a directed path from v to q, because $q \in R(v)$. This also means there is a directed path from q to u, as $q \in W(u)$. Therefore, by appending the directed path from v to q with the directed path from q to u, one obtains a directed path from v to u. This contradicts our assumption that the graph is not strongly connected and there is no directed path

from v to u.

5. Observe that there may be edges originating in $W(u)$ and terminating in either $R(v)$ or $Q(u,v)$. However, there can be no edges originating in $R(v)$ or $Q(u,v)$ and terminating in $W(u)$. Similar observations can be made with other set combinations.

Each node in the graph $\mathcal{G}(M)$ corresponds to a row/column of the matrix M. Therefore, we can construct a permutation matrix P such that the first rows of P^TMP correspond to the nodes in $W(u)$, the next set of rows correspond to the nodes in $Q(u,v)$, and the last set of rows to the nodes in $R(v)$.

$$
\begin{aligned}
\overline{M} = P^TMP &= \quad W(u) \mid Q(u,v) \mid R(v) \\
&= \begin{array}{c|c}
\left[\begin{array}{c|c|c}
\overline{M}_{W,W} & \overline{M}_{W,Q} & \overline{M}_{W,R} \\ \hline
\overline{M}_{Q,W} & \overline{M}_{Q,Q} & \overline{M}_{R,R} \\ \hline
\overline{M}_{R,W} & \overline{M}_{R,Q} & \overline{M}_{R,R}
\end{array}\right] & \begin{array}{c} W(u) \\ \hline Q(u,v) \\ \hline R(v) \end{array}
\end{array}
\end{aligned}
$$

The matrix $\overline{M}$ can be expressed as a 3×3 block matrix. Each block will therefore encode certain properties about the associated directed graph. For example, the block $\overline{M}_{w,w}$ can be used to describe the induced sub-graph containing only the nodes in the set $W(u)$. Based on how the sets were defined, we can arrive at some conclusions about the structure of some of these blocks. In particular, if we can show that blocks in the lower left-corner are identically zero, then we can conclude this part.

Let us consider the block $\overline{M}_{R,W}$. There can only be a non-zero entry in this block if there exists an edge from a node in $R(v)$ to a node in $W(u)$. However, this contradicts the assumption that there is no directed path from node v to u (using the same argument from item 4 in the above list). Therefore, we can conclude that $\overline{M}_{r,w} = \mathbf{0}$.

In this way, we can also conclude that $\overline{M}_{Q,W} = \mathbf{0}$ and $\overline{M}_{R,Q} = \mathbf{0}$. Note that the other blocks may or may not be equal to zero, depending on the particular graph. Therefore, using our specified permutation matrix, we have that

$$
\overline{M} = \left[\begin{array}{c|c|c}
\overline{M}_{W,W} & \overline{M}_{W,Q} & \overline{M}_{W,R} \\ \hline
\mathbf{0} & \overline{M}_{Q,Q} & \overline{M}_{Q,R} \\ \hline
\mathbf{0} & \mathbf{0} & \overline{M}_{R,R}
\end{array}\right].
$$

Using this description, we can now see that

$$
\overline{M} = \begin{bmatrix} B & C \\ \mathbf{0} & D \end{bmatrix},
$$

with

$$
\begin{aligned}
B &= \overline{M}_{W,W} \in \mathbb{R}^{|W(u)| \times |W(u)|} \\
C &= \begin{bmatrix} \overline{M}_{W,Q} & \overline{M}_{W,R} \end{bmatrix} \in \mathbb{R}^{|W(u)| \times |Q(u,v) \cup R(v)|} \\
D &= \begin{bmatrix} \overline{M}_{Q,Q} & \overline{M}_{Q,R} \\ \mathbf{0} & \overline{M}_{R,R} \end{bmatrix} \in \mathbb{R}^{|Q(u,v) \cup R(v)| \times |Q(u,v) \cup R(v)|}
\end{aligned}
$$

The structure of $\overline{M}$ is therefore reducible, with $r = |W(u)|$ and $n = |mcV(M)|$. This contradicts the original statement that M is irreducible, concluding the first part of the proof.

M is irreducible $\Leftarrow \mathcal{G}(M)$ is strongly connected

The converse can be proved in the same way. As a sketch, assume that M is reducible. This implies there is a permutation matrix P such that $P^T MP$ takes the form of a reduced matrix. One can then use this to argue that there are subsets of nodes in the graph where a directed path can not be found connecting them.