Exercise #1

Analysis and Control of Multi-Agent Systems

Problem 1

i) ***Mesbahi & Egerstedt*** Exercise 2.2

The degree sequence for a graph is a listing of the degrees of its nodes; thus K_3 has the degree sequence 2,2,2. Is there a graph with the degree sequence 3,3,3,3,5,6,6,6,6,6,6? How about with the degree sequence 1,1,3,3,3,3,5,6,8,9? If yes, draw the graph; otherwise, explain why there is no such graph.

ii) ***Mesbahi & Egerstedt*** Exercise 2.5

Let $\mathcal{G}$ be an undirected graph on n vertices with c connected components. Show that $\operatorname{rank} L(\mathcal{G}) = n - c$.

iii) ***Mesbahi & Egerstedt*** Exercise 2.11

Show that any graph on n vertices that has more than $n-1$ edges contains a cycle.

iv)

Explain clearly what is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3. Explain why this is the maximum value.

v)

For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, one can define the *complement graph*, $\overline{\mathcal{G}} = (\mathcal{V}, \overline{\mathcal{E}})$, with $\overline{\mathcal{E}} = [\mathcal{V}]^2 \setminus \mathcal{E}$.

- Show that $\mathcal{G} \cup \overline{\mathcal{G}} = K_n$.
- Show that $\mathcal{G}$ and $\overline{\mathcal{G}}$ can not both be disconnected.

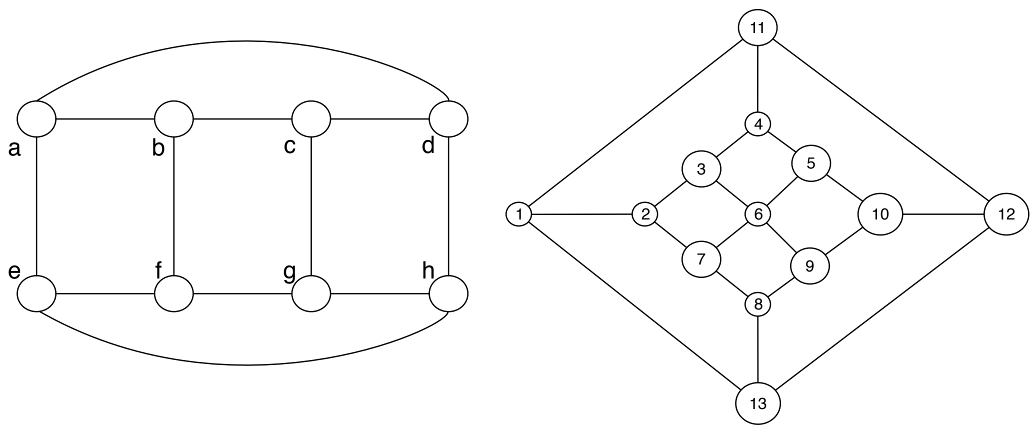
Problem 2

Bipartite Graphs

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *bipartite* if there is a partition of the node-set $\mathcal{V}$ into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ (that is, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$), such that every edge connects a vertex in $\mathcal{V}_1$ to a vertex in $\mathcal{V}_2$ (i.e., $\mathcal{E} \subseteq \{\{v_i, v_j\} \,|\, v_i \in \mathcal{V}_1, v_j \in \mathcal{V}_2\}$).

i)

Determine whether the following graphs are bipartite. If so, give the partition of the vertices into two sets.



(a) Graph 1 (b) Graph 2

Figure 1: Are these graphs bipartite?

ii)

The adjacency matrix of a bipartite graph $\mathcal{G}$ can always be expressed (after a relabeling of the nodes) as

$$A(\mathcal{G}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where $A(\mathcal{G}) \in \mathbb{R}^{n_1+n_2 \times n_1+n_2}$ and $B \in \mathbb{R}^{n_1 \times n_2}$. Show that if λ is an eigenvalue of $A(\mathcal{G})$ then so is $-\lambda$.

iii)

Prove that a connected k-regular graph $\mathcal{G}$ is bipartite if and only if $-k$ is an eigenvalue of $A(\mathcal{G})$.

Problem 3

7 Bridges of Königsberg

The city of Königsberg in Prussia was split by the Pregel river. The city also included two large islands and were connected to the mainland with a network of 7 bridges; see Figure 2. In 1735, Leohnard Euler set out to solve the problem of whether or not there was a walk through the city that crossed each of the 7 bridges exactly once. The basis of his solution is considered to be the origin of modern graph Theory.

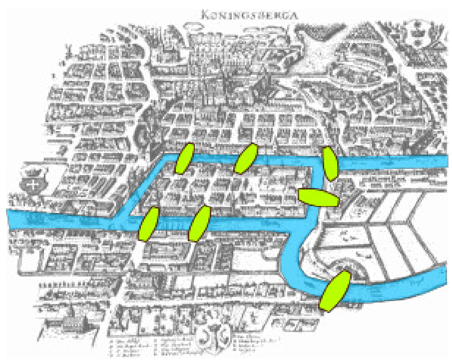


Figure 2: Map of Königsberg in 1735.

Using the map in Figure 2, draw an equivalent graph-theoretic representation. That is, define a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that represents the network of bridges and how it connects the land masses of Königsberg. Determine the answer to the seven bridges problem.

An *Eulerian Circuit*, named in honor of Leohnard Euler, is a walk in a graph beginning and ending at the same vertex that uses every edge exactly once. Prove that a necessary condition for a graph to have an Eulerian Circuit (sometimes called an *Eulerian graph*) is that the degree of each node must be even. How does this relate to the seven bridges of Königsberg problem?

Problem 4

Incidence Matrix

Consider the following incidence matrix,

$$E(\mathcal{G}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

i)

Draw the associated graph and find a spanning-tree subgraph.

i)

Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}_\mathcal{T}) \subset \mathcal{G}$ be a spanning-tree subgraph and $E(\mathcal{T})$ its associated incidence matrix. The edges $\mathcal{E}_\mathcal{C} = \mathcal{E} \setminus \mathcal{E}_\mathcal{T}$ are the remaining edges that are not in the spanning tree. Let $\mathcal{C} = (\mathcal{V}, \mathcal{E}_\mathcal{C}) \subset \mathcal{G}$ be the subgraph containing all the edges not included in the spanning tree with incidence matrix $E(\mathcal{C})$.

Find an algebraic relationship between $E(\mathcal{T})$ and $E(\mathcal{C})$. Show that the null-space of the incidence matrix, $\text{null}\{E(\mathcal{G})\}$, is related to the independent cycles in $\mathcal{G}$.

Problem 5

Adjacency Matrix

Prove the following results on the adjacency matrix:

Lemma 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with adjacency matrix $A(\mathcal{G})$. The number of walks from node v_i to v_j of length r is $[A(\mathcal{G})^r]_{ij}$ (for a matrix $M \in \mathbb{R}^{n \times m}$, $[M]_{ij}$ refers to the element in the ith row and jth column of M).*

Corollary 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with e edges, t triangles (cycle of length 3), and adjacency matrix $A(\mathcal{G})$. Then*

- $\mathbf{trace} A(\mathcal{G}) = 0$,
- $\mathbf{trace} A(\mathcal{G})^2 = 2e$,
- $\mathbf{trace} A(\mathcal{G})^3 = 6t$.

Problem 6

Matrix-Tree Theorem

Consider the graph in Figure 3

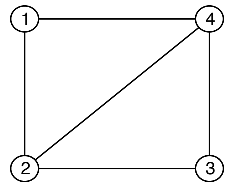


Figure 3: A graph on 4 nodes.

i)

Write the Laplacian matrix $L(\mathcal{G})$ associated with graph in Figure 3. Enumerate all the spanning trees and verify that the number of spanning trees can be computed from the determinant of any sub-matrix of $L(\mathcal{G})$ obtained by deleting a row and column.