Exercise #2

Analysis and Control of Multi-Agent Systems

Problem 1

(*Mesbahi & Egerstedt 3.1, 3.2, 3.8*)

Exercise 3.1

Simulate the agreement protocol for an undirected graph on five vertices. Compare the rate of convergence of the protocol as the number of edges increases. Does the convergence rate always improve when the graph contains more edges? Provide a formal analysis to support your observation.

Exercise 3.2

Consider the digraph $\mathcal{D}$ and the following symmetric protocol,

$$\dot{x}(t) = \frac{1}{2}\left(L_{in}(\mathcal{D}) + L_{in}(\mathcal{D})^T\right)x(t).$$

Does this protocol correspond to the agreement protocol on a certain graph? What are the conditions on the digraph $\mathcal{D}$ such that the resulting protocol converges to the agreement subspace?

Exercise 3.8

Consider the uniformly delayed agreement dynamics over a weighted graph, specified as

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}(i)} w_{ij}(x_j(t-\tau) - x_i(t-\tau)),\ i = 1,\ldots,n,$$

for some $\tau > 0$. Show that this delayed protocol is stable if

$$\tau < \frac{\pi}{2\lambda_n(\mathcal{G})},$$

where $\lambda_n(\mathcal{G})$ is the largest eigenvalue of the corresponding weighted Laplacian. Conclude that for the *delayed agreement protocol*, there is a trade-off between faster convergence rate and tolerance to uniform delays on the information-exchange links.

Hint: Consider the following auxiliary system:

$$\dot{x}(t) = -L_w(\mathcal{G})x(t-\tau) + u,$$

where $u \in \mathbb{R}^n$ is an arbitrary constant vector. Determine the transfer-function of the system recalling that following Laplace transform identity:

$$x(t-\tau) \xrightarrow{\mathcal{L}} e^{-\tau s}X(s).$$

Find the smallest valley for τ that will place a pole on the $j\omega-$axis.

Problem 2

Bounded Confidence

The consensus protocol is a *linear* protocol. Consider the following non-linear state-dependent consensus protocol,

$$\dot{x}_i(t) = \sum_{x_j \in \mathcal{N}_{v_i}(t)} (x_j(t) - x_i(t))$$

where $\mathcal{N}_{v_i}(t) = \{v_j \in \mathcal{V} \,|\, |x_i(t) - x_j(t)| < 1\}$. That is, agent i only senses agent j if they are less than one unit apart. Simulate this non-linear protocol for a variety of different initial conditions. Also include initial conditions that are uniformly distributed on the interval $[0\ n]$ where n is the number of agents. What phenomena do you observe? Conjecture a necessary condition for such a system to reach agreement.

This model is known as the *bounded confidence opinion dynamics* model, originally developed by Ulrich Krause. It is used to model the evolution of opinions in certain social networks.

Problem 3

Eigenvalues of Laplacian Matrix

Show/Prove the following bounds on the eigenvalue of the undirected and unweighted graph Laplacian matrix.

1. $\lambda_n(\mathcal{G}) \geq \max_i d_i$
2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\hat{G} = (\mathcal{V}, \mathcal{E} \cup \{\hat{e}\})$. Prove that $\lambda_2(\mathcal{G}) \leq \lambda_2(\hat{\mathcal{G}}) \leq \lambda_2(\mathcal{G}) + 2$

Hint: For the first part, recall the variational characterization of the eigenvalue. For example,

$$\lambda_n(\mathcal{G}) = \max_{\|x\|=1} x^T L(\mathcal{G}) x.$$

What is the variational characterization for $\lambda_2(\mathcal{G})$? For part 2, think about how to write the Laplacian of $\hat{G}$ in terms of $L(\mathcal{G})$ plus an additional term.

Problem 4

Agreement Over Directed Graphs

1. Consider the agreement protocol over a weighted digraph $\mathcal{G}$,

   $$\dot{x}(t) = L_{in}(\mathcal{G}) x(t),$$

   and assume that $\mathcal{G}$ contains a rooted out-branching. Is there a 'constant of motion' for this protocol? Provide an analysis to justify your answer.

2. Can the agreement protocol over any weighted digraph be expressed as a gradient dynamical system? If so, what is the appropriate function that defines the gradient dynamics? If not, are there certain classes of weighted digraphs that can be expressed as a gradient dynamical system?