Exercise #3

Analysis and Control of Multi-Agent Systems

Problem 1

(*Mesbahi & Egerstedt 3.9, 3.11, 10.1, 10.5*)

Exercise 3.9

A matrix M is called *essentially non-negative* if there exists a sufficiently large μ such that $M + \mu I$ is non-negative, that is, all its entries are non-negative. Show that e^{Mt} for an essentially non-negative matrix M is non-negative when $t \geq 0$.

Exercise 3.11

Consider vertex i in the context of the agreement protocol $\dot{x} = -L(\mathcal{G})x$. Suppose that vertex i (the rebel) decides not to abide by the agreement protocol, and instead fixes its state to a constant value (i.e., $x_i(t) = c$ for all $t \geq 0$). Show that all vertices converge to the state of the rebel vertex when the graph is connected.

Exercise 10.1

Consider a connected, undirected network with input nodes (one or more). Let the floating nodes be running the standard agreement protocol. Show that if the network is not controllable, then the uncontrollable part of the system is asymptotically stable.
Note: This problem refers to the controlled agreement protocol. We assume here that there may be one or more control (anchor) nodes, and the rest of the graph (the follower graph, or floating nodes) follows the standard agreement.

Exercise 10.5

Given an input network and assume that the input nodes' positions can be controlled directly while the floating nodes' dynamics satisfy the agreement protocol. With this setup, consider the networks below, where the input nodes are given in black, and the floating nodes in white. Which (if any) of the networks are controllable?

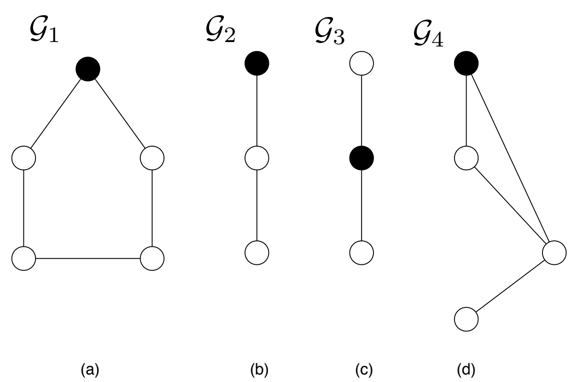

Figure 1: Which configurations are controllable?

1.5) Prove that the matrix $R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T$ (where $R_{(\mathcal{T},\mathcal{C})} = [I \ T_{(\mathcal{T},\mathcal{C})}]$) is invertible. Provide graph-theoretic interpretations for the entries of the matrix $T_{(\mathcal{T},\mathcal{C})}T_{(\mathcal{T},\mathcal{C})}^T$ and $T_{(\mathcal{T},\mathcal{C})}^T T_{(\mathcal{T},\mathcal{C})}$.

Controlled Agreement

Problem 2

Consider the controlled agreement protocol studied in lecture. Prove the following theorem:

Theorem 1. *The controlled agreement protocol is uncontrollable if it is input symmetric. Equivalently, the controlled agreement protocol is uncontrollable if the floating graph admits a nonidentity automorphism for which the input indicator vector remains invariant under its action.*

Hint: Assume that the eigenvalues of A_f are unique (why should we do this?). Beginning with the assumption of input symmetry, show that if v is an eigenvector of A_f that Jv is also an eigenvector (with J a permutation matrix). Use this fact and the other algebraic conditions for controllability to show controlled agreement protocol is uncontrollable.

Problem 3

Incidence Matrix and Edge Laplacian

1

Given the following incidence matrix,

$$E(\mathcal{G}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},$$

draw the associated graph and compute the matrix T. How many cycles are in the graph? Are all the cycles "linearly independent?" Devise an algorithm that constructs the matrix T without computing any matrix inverses.

2

1. Prove that the non-zero eigenvalues of $L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$ are the same as the non-zero eigenvalues of $L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$.

2. Characterize the null-space of the edge Laplacian $L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$.

3. Find a similarity transformation matrix between the matrix $L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$ and the matrix

$$\begin{bmatrix} L(\mathcal{G}) & 0 \\ 0 & \mathbf{0} \end{bmatrix},$$

where the matrix $\mathbf{0}$ is a square matrix of appropriate size of all zeros (i.e. a matrix S such that $S^{-1}L_e(\mathcal{G})S$ is the matrix shown above). What should the size of that zero-block matrix be and what graph theoretic property is it related to?

Problem 4

Edge Agreement

4.1) Consider the consensus protocol over a connected graph $\mathcal{G}$ with spanning tree $\mathcal{T}$ corrupted by both process noise $w(t)$ at the input, and sensor noises $v(t)$ on the relative outputs, i.e.,

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) &= u(t) + w(t) \\ y(t) &= E(\mathcal{G})^T x(t) + v(t) \\ z(t) &= E(\mathcal{T})^T x(t) \end{cases},$$

and $u(t) = -E(\mathcal{G})y(t)$. What are the closed-loop dynamics for the above system? What is the associated edge agreement protocol for this system?

4.2) For the edge agreement protocol found above, find an expression for the $\mathcal{H}_2$ performance when the controlled output is $z(t) = E(\mathcal{T})^T x(t)$ and for when it is $z(t) = E(\mathcal{G})^T x(t)$. Comment on the difference.)

4.3) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph obtained by adding two edges to the spanning tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}_\tau)$; that is, $\mathcal{E} = \mathcal{E}_\tau \cup \{e_1, e_2\}$ with $e_1, e_2 \notin \mathcal{E}_\tau$. Consider now the edge agreement protocol over the graph $\mathcal{G}$ with only process noises, i.e.,

$$\Sigma_e(\mathcal{G}) : \begin{cases} \dot{x}_\tau(t) &= -L_e(\mathcal{T}) R_{(\mathcal{T},\mathcal{C})} R_{(\mathcal{T},\mathcal{C})}^T x_\tau(t) + E(\mathcal{T})^T w(t) \\ z(t) &= x_\tau(t) \end{cases}.$$

Derive an expression for the $\mathcal{H}_2$ performance and provide a graph-theoretic interpretation of the result

Hint: Use the solution to exercise 1.5 to help with the interpretation of the result.

4.4) Consider the consensus protocol over a connected graph $\mathcal{G}$ corrupted by process noises. We would like to examine the $\mathcal{H}_2$ performance of the edge agreement protocol for two different possible performance output variables:

$$\begin{aligned} z_1(t) &= E(\mathcal{T}_1)^T x(t) \\ z_2(t) &= E(\mathcal{T}_2)^T x(t), \end{aligned}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are two different spanning trees of $\mathcal{G}$. Assume that in both cases, the state equation is

$$\dot{x}_\tau = -L_e(\mathcal{T}_1) R_{(\mathcal{T}_1,\mathcal{C})} R_{(\mathcal{T}_1,\mathcal{C})}^T x_\tau(t) + E(\mathcal{T}_1)^T w(t).$$

How is the $\mathcal{H}_2$ performance of the corresponding edge agreement problem for each performance variable related?