Exercise #4

Analysis and Control of Multi-Agent Systems

Problem 1

Relative Sensing Networks

1.1) Consider a collection of n agents with identical dynamics,

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bw_i(t) \\ y_i(t) &= Cx_i(t), \end{aligned}$$

for $i = 1, \ldots, n$. The output of the networked system are the relative outputs of neighboring agents as defined by a sensing graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. That is, the output is the vector $z(t) \in \mathbb{R}^{|\mathcal{E}|}$ where $z_k(t) = y_i(t) - y_j(t)$ for $k = (i, j) \in \mathcal{E}$. Express the collective dynamics of the entire team in state-space form.

1.2) What is the $\mathcal{H}_2$ performance of the homogeneous RSN above? Among the set of all connected graphs, what class of graphs are $\mathcal{H}_2$ optimal?

1.3) Recall that the $\mathcal{H}_2$ performance of the heterogeneous RSN is $\|\Sigma(\mathcal{G})\|_2^2 = \sum_{i=1} d_i \|\Sigma_i\|_2^2$. Show that the norm can be expressed as

$$\|\Sigma(\mathcal{G})\|_2^2 = \|WE(\mathcal{G})\|_F^2$$

for an appropriately defined diagonal matrix W.

1.4 Consider a heterogeneous relative sensing network where the sensing graph $\mathcal{G}$ is generically infinitesimally, and minimally rigid and has $\mathcal{H}_2$ performance $\|\Sigma(\mathcal{G})\|_2$. Suppose there is a new agent Σ_o that joins the relative sensing network. Propose a procedure for adding the new agent to the network such that

1. **The new relative sensing network is still generically infinitesimally and minimally rigid,**
2. **The $\mathcal{H}_2$ performance of the new RSN is increased by the smallest amount possible.**

Problem 2

Minimum Weight Spanning tree

2.1) For the weighted graph in Figure 1, find the minimum weight spanning tree using Kruskal's Algorithm. Implement Kruskal's Algorithm in MATLAB (or similar).

2.2) Provide conditions on the weights that guarantee a *unique* solution to the minimum weight spanning tree algorithm. Construct an example of a weighted graph that does not admit a unique solution to the minimum weight spanning tree problem.

2.3) Consider the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. Formulate the minimum weight spanning tree problem as an *integer program*.

Hint: Let $x_e \in \{0, 1\}$ be the binary decision variable to include edge $e \in \mathcal{E}$ in the minimum weight tree. First, express the objective function in terms of the decision variables x_e. The integer program should include linear

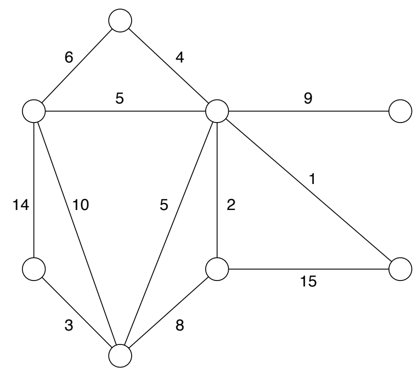

Figure 1: Find the minimum weight spanning tree

constraints that reflect all the required properties of a spanning tree. The challenging part is to find a linear set of constraints that guarantees there are no cycles!

Problem 3

Graph Rigidity

3.1 Show that the framework $(\mathcal{G}, p_0)$ in Figure 2 is rigid but not infinitesimally rigid.

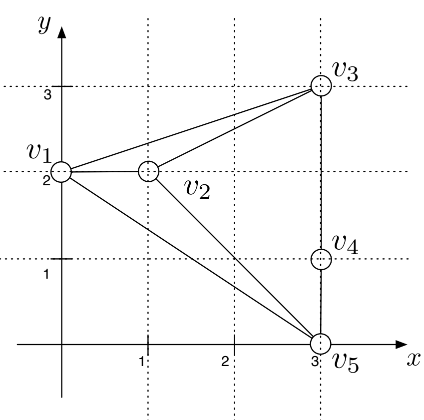

Figure 2: A framework $(\mathcal{G}, p_0)$ in $\mathbb{R}^2$.

3.2 Show that adding the edge $\{v_2, v_4\}$ to create the graph $\mathcal{G}' = (\mathcal{V}, \mathcal{E} \cup \{v_2, v_4\})$ results in the framework $(\mathcal{G}', p_0)$ being infinitesimally rigid. Is the resulting graph generically rigid?

3.3 Show how the framework in Figure 2 can be generated using the Henneberg Construction. Can the framework in Figure 2 with the added edge $\{v_2, v_4\}$ also be constructed by a Henneberg Construction? If yes, show the construction, and if not, explain why.

Problem 4

Formation Control

Consider the distance-based formation control law derived in class,

$$\dot{p}_i = \sum_{j \sim i} \left(\|p_i - p_j\|^2 - d_{ij}^2 \right) (p_j - p_i).$$

Here we assume that $p_i(t) \in \mathbb{R}^2$.

1. Simulate the protocol for a team of three agents with the desired formation an equilateral triangle for a variety of initial conditions. Find a set of initial conditions where the protocol does not converge to the desired formation.
2. Simulate the protocol for a team of 5 agents to a desired formation of your choice. Simulate for frameworks that are infinitesimally rigid, minimally infinitesimally rigid, not rigid with full row-rank rigidity matrix, and not rigid with rigidity matrix that is not full row rank.
3. Prove that the centroid of the formation, $\overline{p} = \frac{1}{n}\sum_{i=1}^{|\mathcal{V}|} p_i$, is an invariant quantity (i.e., a constant of motion) for the distance-based formation control.
4. Prove that the set $M_q = \{p = \left[\begin{array}{ccc} p_1 & \cdots & p_{|\mathcal{V}|}\end{array}\right]^T \in \mathbb{R}^{2|\mathcal{V}|} \,|\, p_i = \alpha_i q,\, \alpha_i \in \mathbb{R}, q \in \mathbb{R}^2, \forall i \in \mathcal{V}\}$ is an invariant set under the dynamics of the distance-based formation control. That is, for any initial configuration $p(0) \in M_q$, one has $p(t) \in M_q$ for all $t \geq 0$.

Formation Control with Collision Avoidance

Consider the following potential function,

$$F(p) = \sum_{i=1}^{|\mathcal{V}|} \sum_{j \sim i} \frac{\left(\|p_i - p_j\|^2 - d_{ij}^2 \right)^2}{\|p_i - p_j\|^2},$$

with $p_i \in \mathbb{R}^2$.

1. Express the associated gradient dynamical system, $\dot{p} = -\nabla F(p)$ in a state-space form. Show that the state-space can be expressed as a *state-dependent weighted Laplacian matrix*. That is,

   $$\dot{p} = -(E(\mathcal{G})W(p)E(\mathcal{G})^T \otimes I_2)p,$$

   where $W(p)$ is an appropriately chosen diagonal matrix with entries that depend on the state p.

2. Characterize the equilibrium configurations of the dynamics. Comment on the equilibrium configurations when the underlying graph is a spanning tree, and when it contains cycles.
3. Simulate the protocol for a spanning tree on 3 nodes and for a cycle on three nodes for a variety of different initial conditions.
4. Prove that when the underlying graph is a spanning tree, the dynamics converge to specified inter agent distances, but not necessarily to the desired 'shape.' Conclude also that neighboring agents will never collide (i.e., $p_i = p_j$).