Analysis and Control of Multi-Agent Systems

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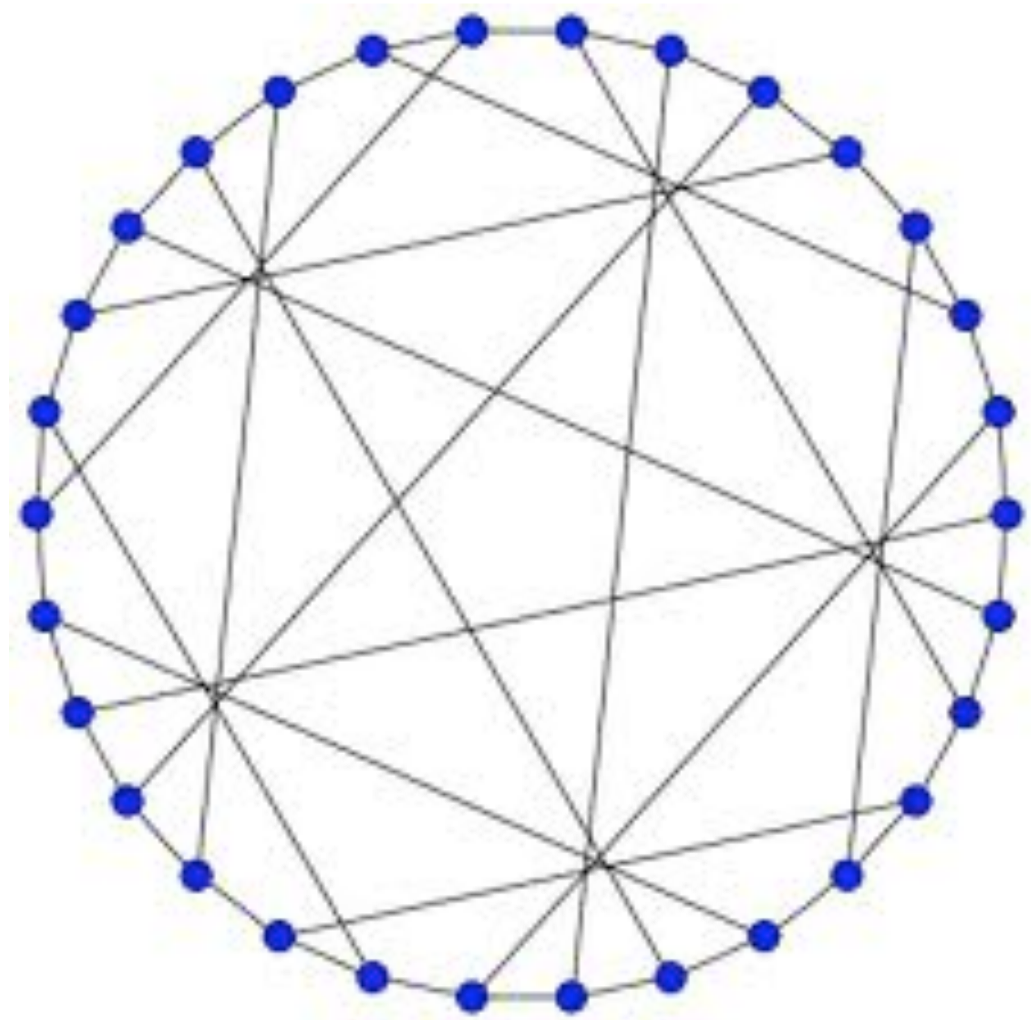
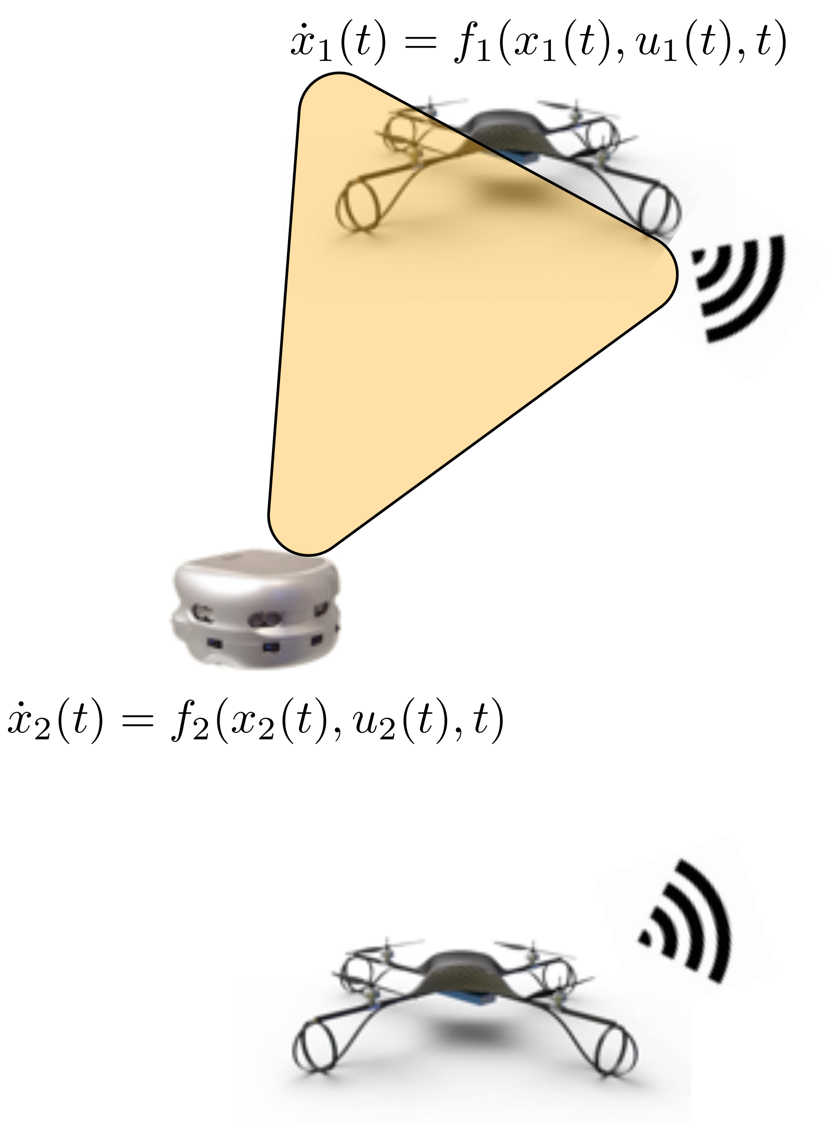
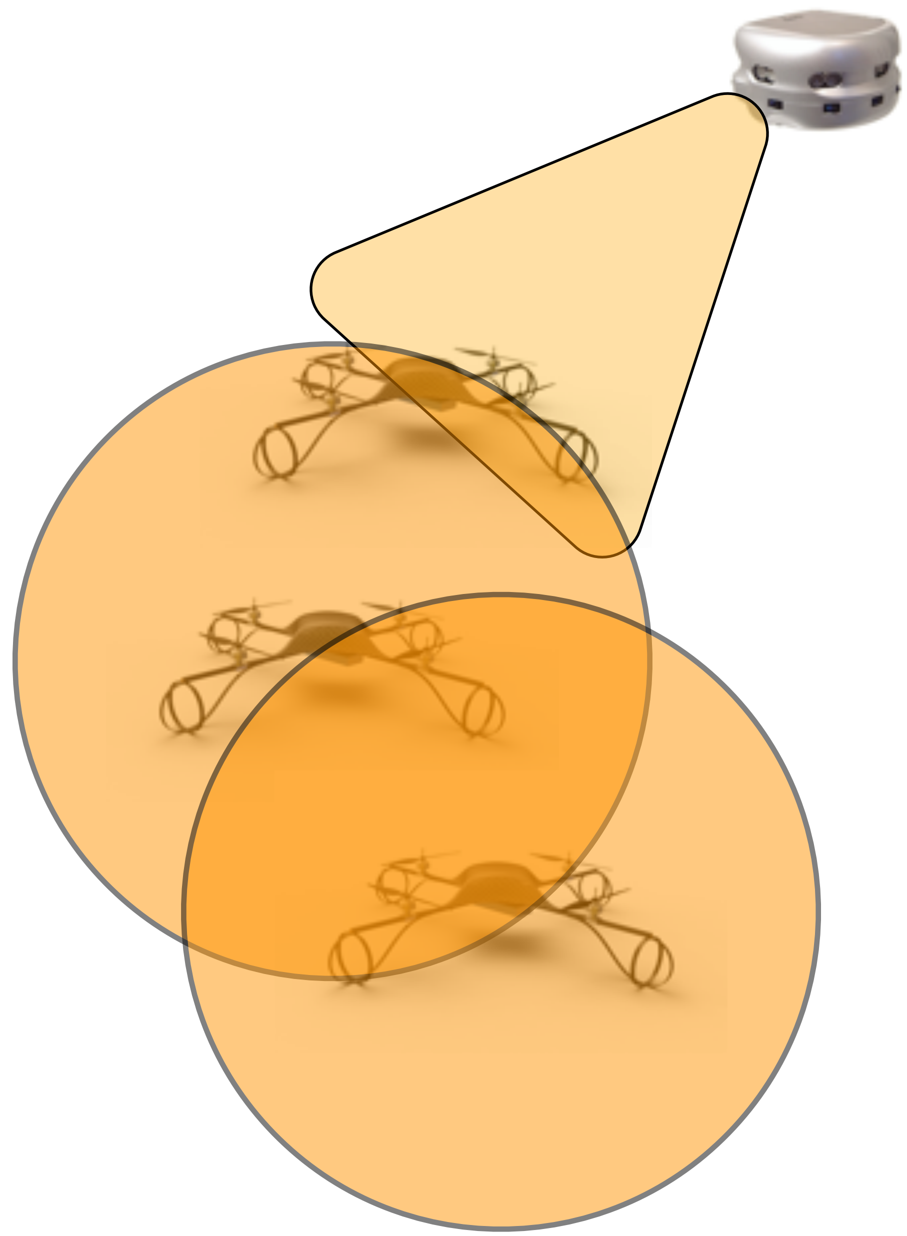
Introduction to Graph Theory

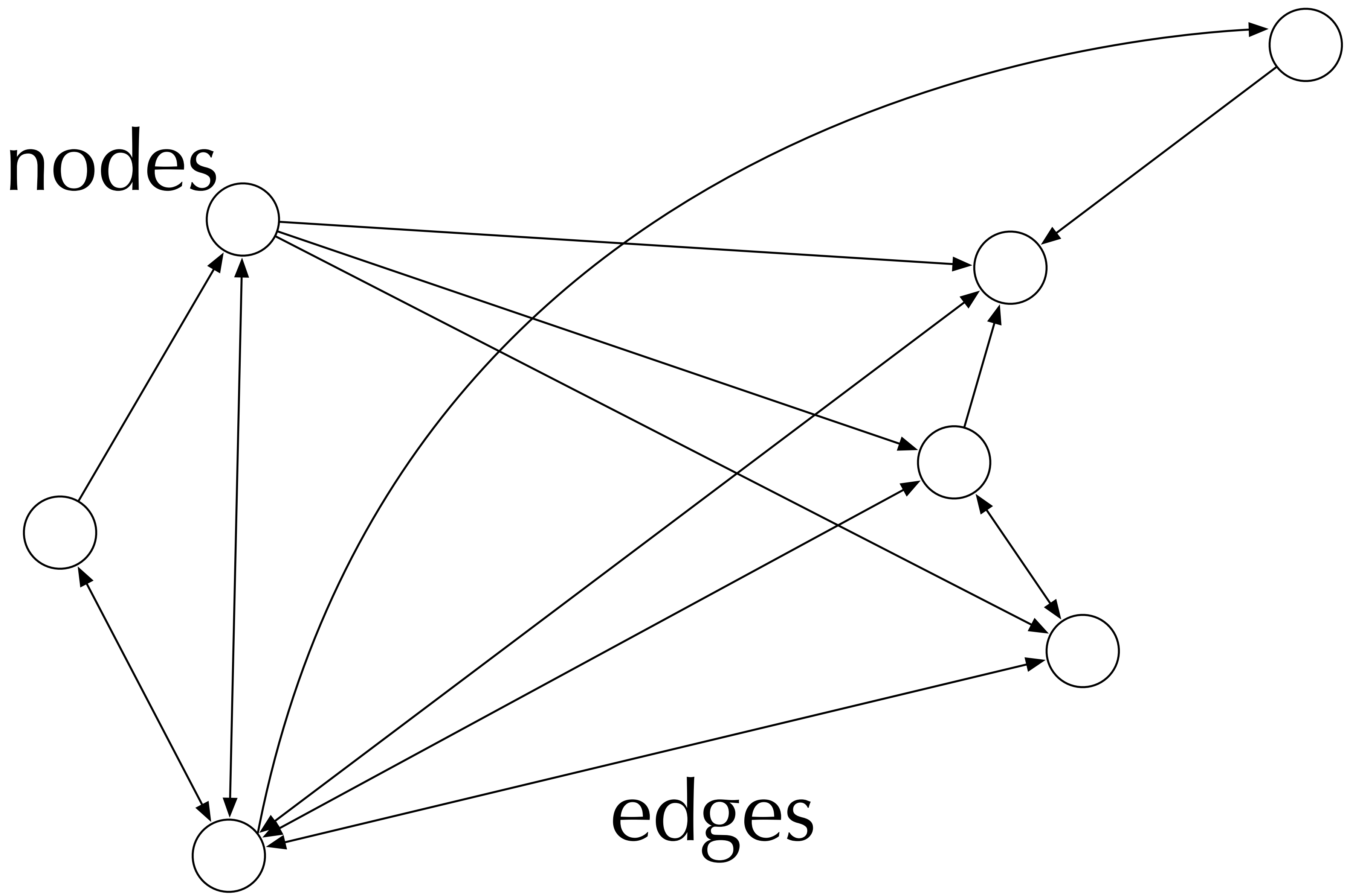


Abstraction Using Graphs



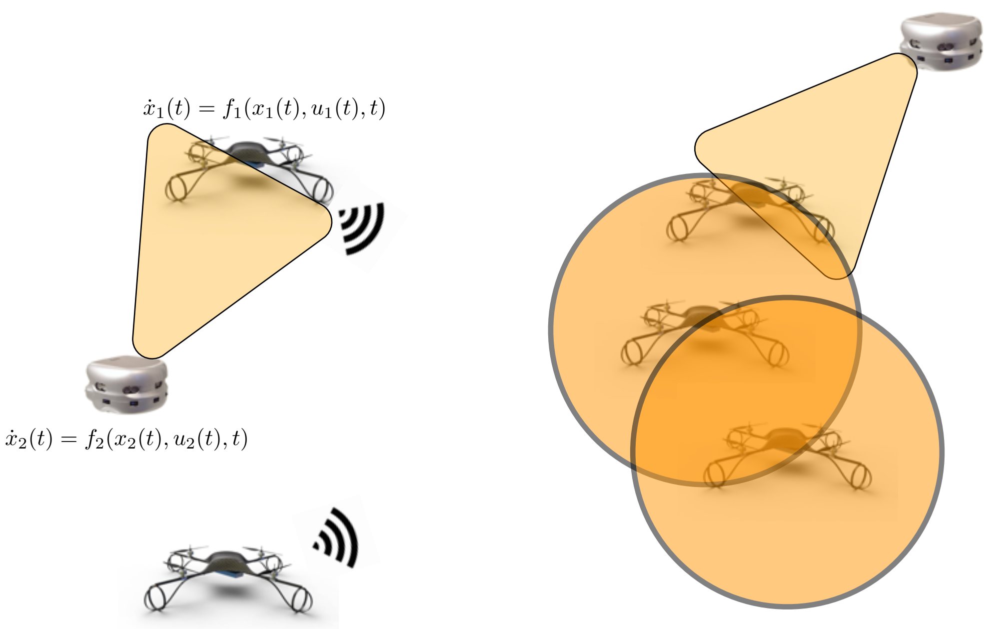


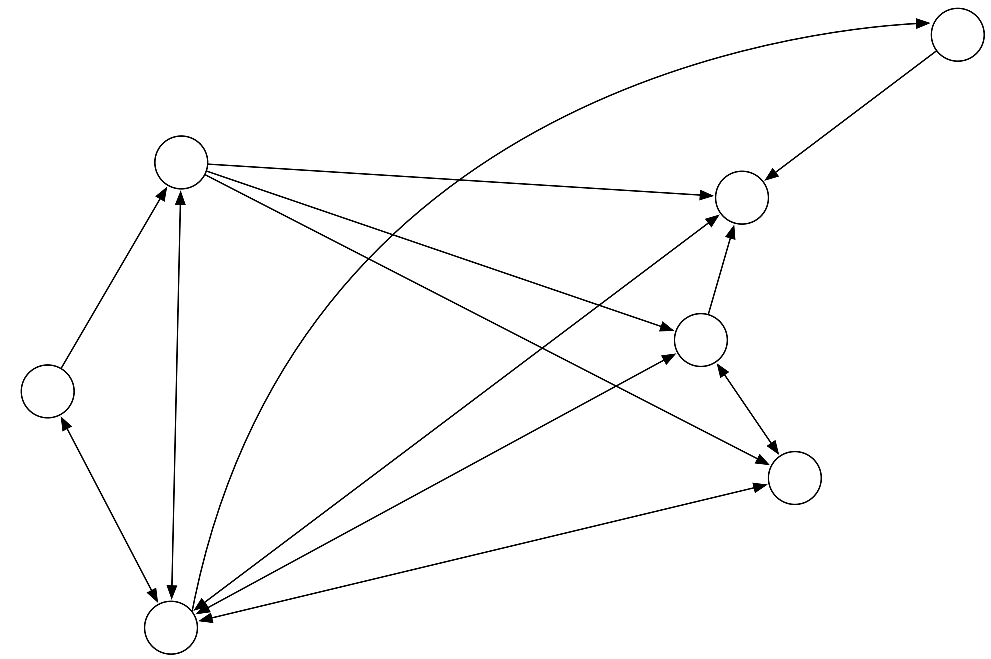
Abstraction Using Graphs



edges can be *directed* or *undirected*

Graph Theory

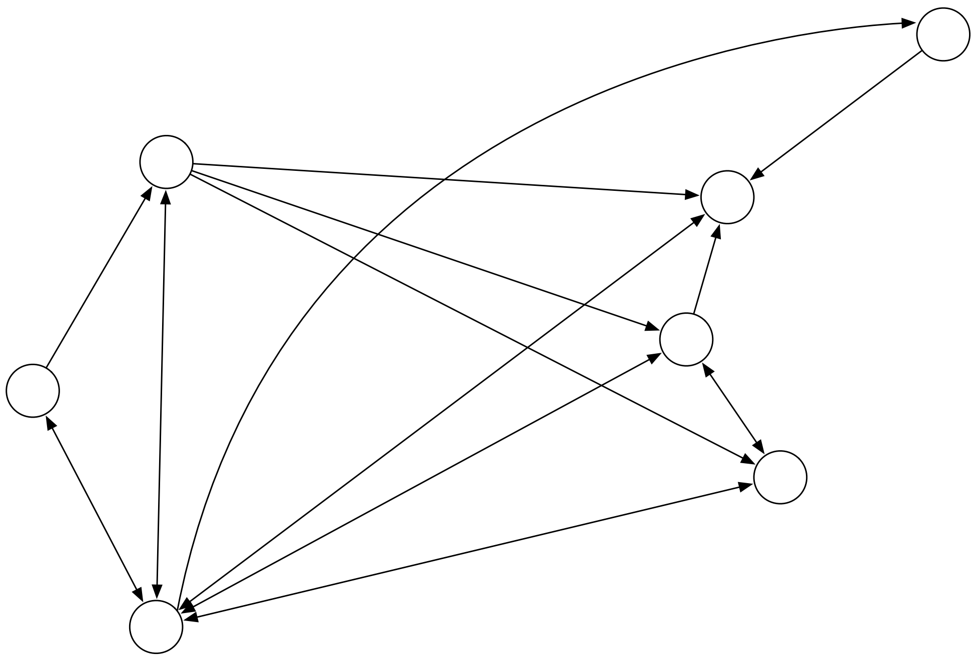




Definition

A ***Graph*** is an ordered pair comprised of a set of **vertices** (or **nodes**), and a set of **edges** (or **links**)

Graph Theory



Notations

a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$

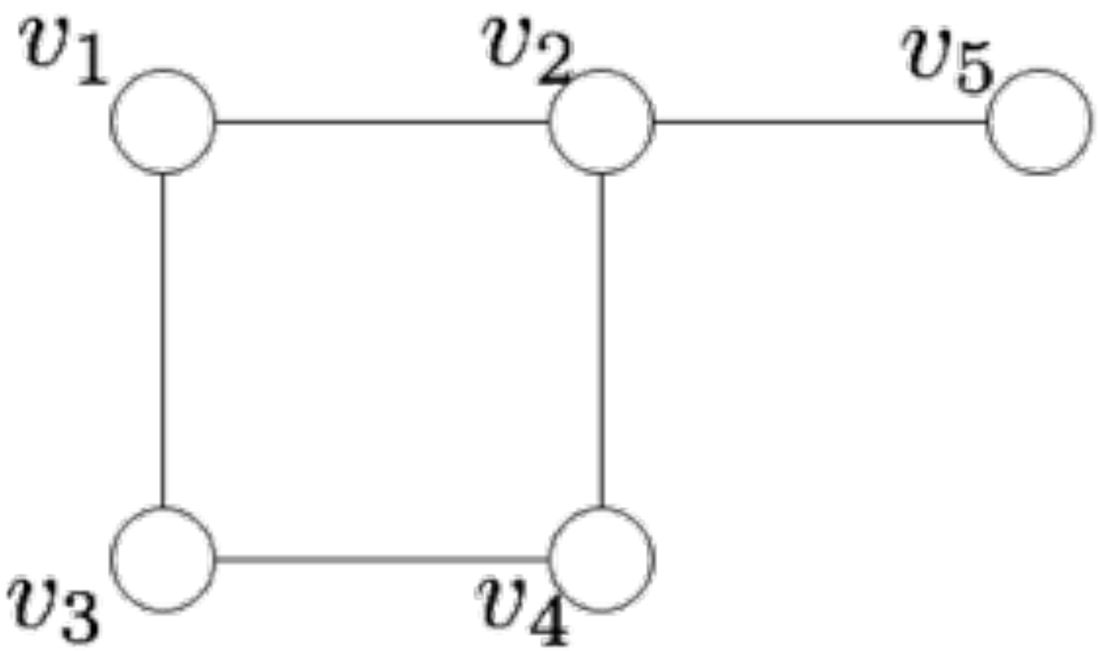
edge set $\mathcal{E} \subseteq [\mathcal{V}]^2$

all 2-element subsets

- undirected graphs
- directed graphs
- weighted graphs

Graph Theory

Example: an *undirected* graph



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

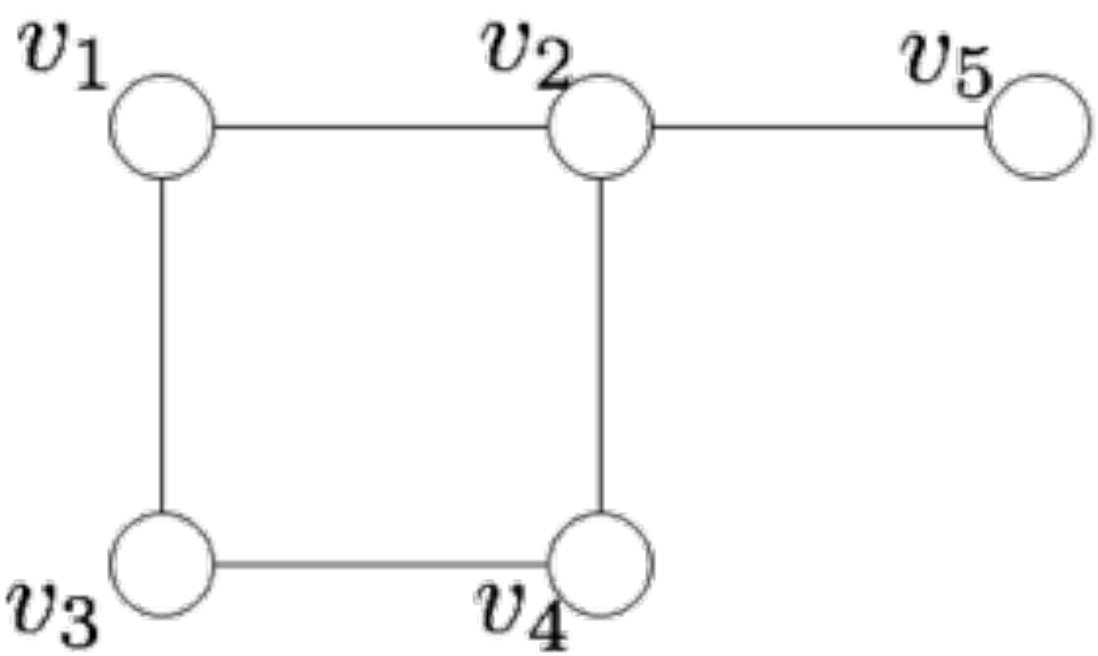
$$\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$$

$$\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}\}$$

$$[\mathcal{V}]^2 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$

Graph Theory

Example: an *undirected* graph



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

more terminology…

- *adjacent* nodes

$$v_1 \sim v_2$$

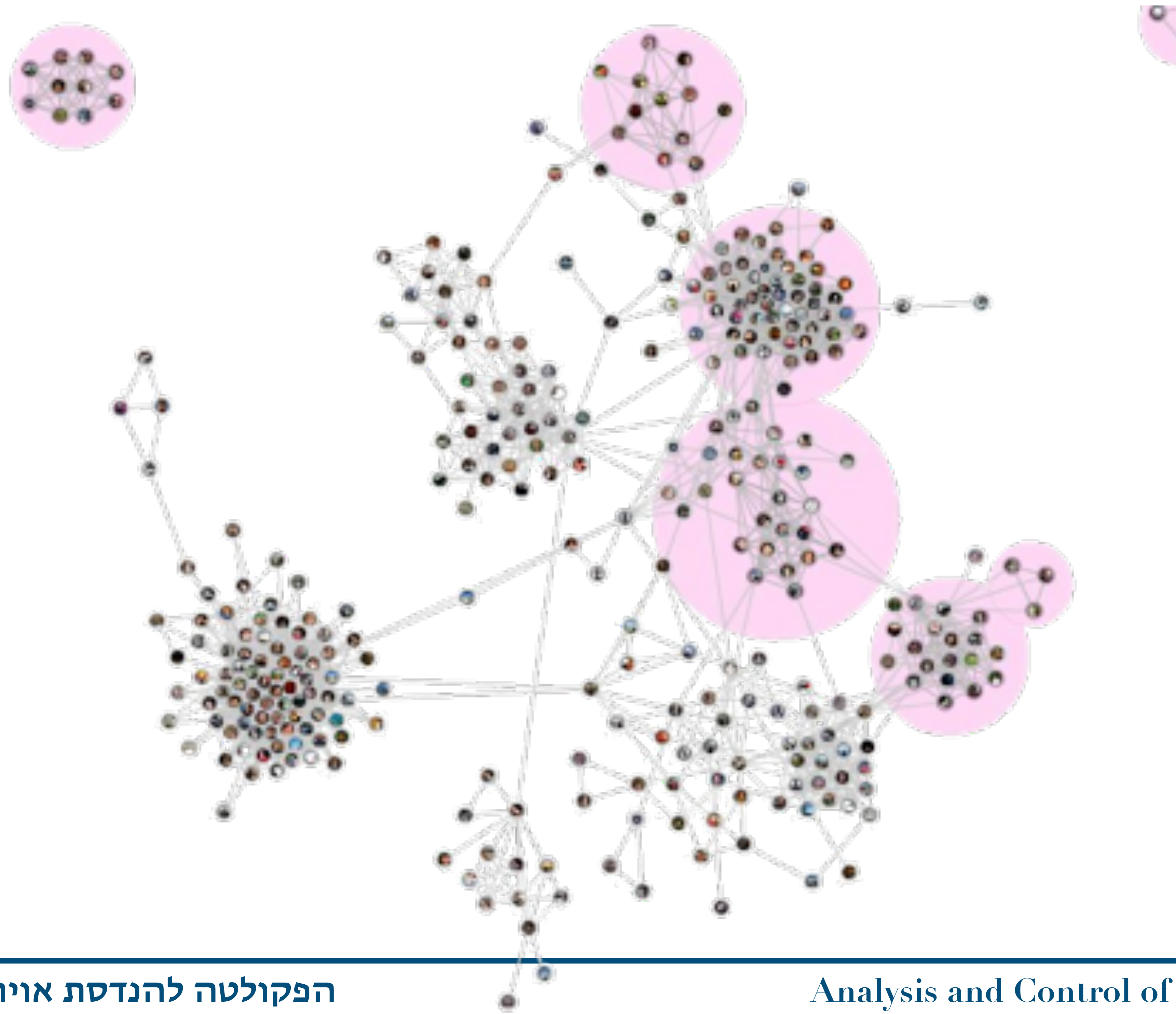
- a node is *incident* to an edge
- *neighborhood*

$$\mathcal{N}(v_i) = \{v_j \in \mathcal{V} \,|\, \{v_i, v_j\} \in \mathcal{E}\}$$

$$\mathcal{N}_{v_i}$$

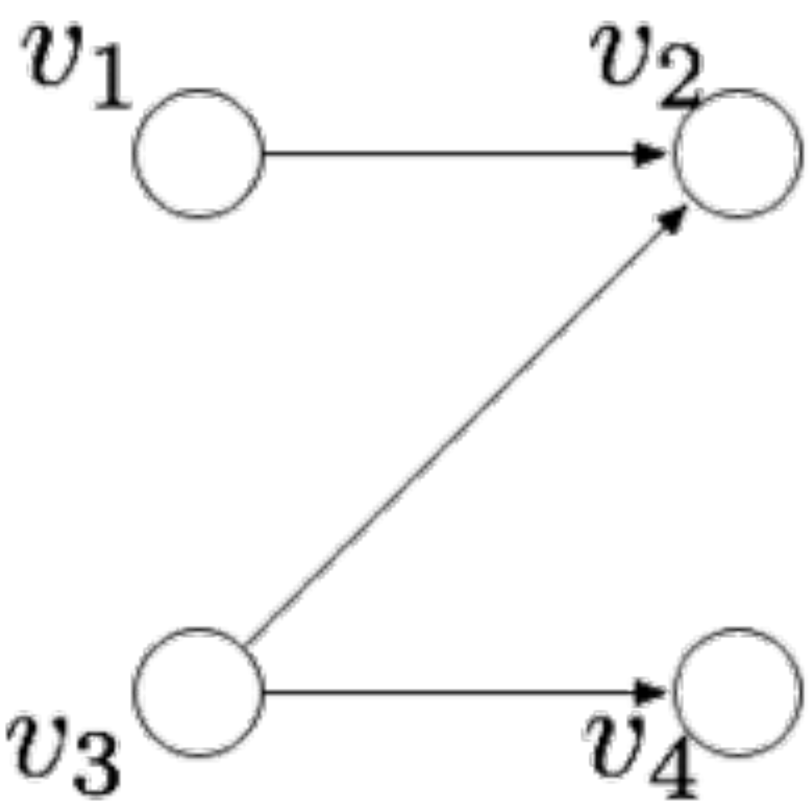
Graph Theory

Example: graphs can model *social interactions*



Graph Theory

Example: a *directed* graph (digraph)



$$\mathcal{V} = \{v_1, v_2, v_3, v_4\}$$

$$\mathcal{E} = \{(v_1, v_2), (v_3, v_2), (v_3, v_4)\}$$

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

- edges are *ordered pairs* with a *head* (*initial*) node and a *tail* (*terminal*) node
- edges are said to have an *orientation*

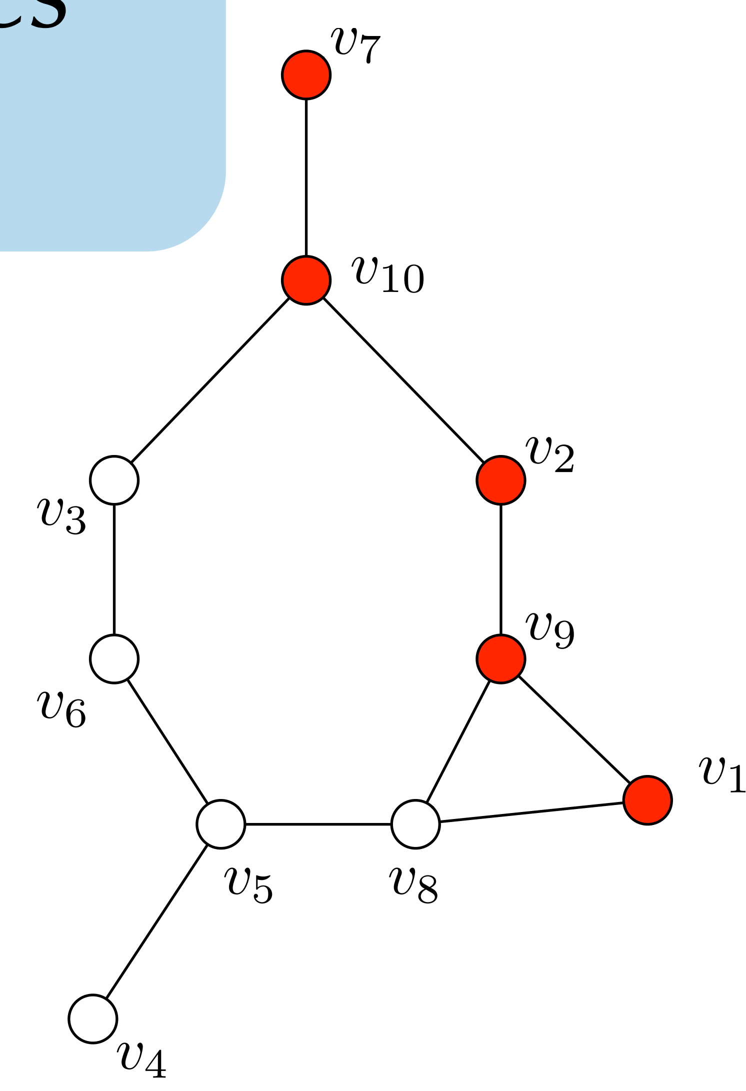
$$\mathcal{D} = (\mathcal{V}, \mathcal{E})$$

Graph Theory

Definition

A (simple) **path** is a sequence of distinct vertices such that consecutive vertices are adjacent.

$$P(v_1, v_7) = v_1 v_9 v_2 v_{10} v_7$$

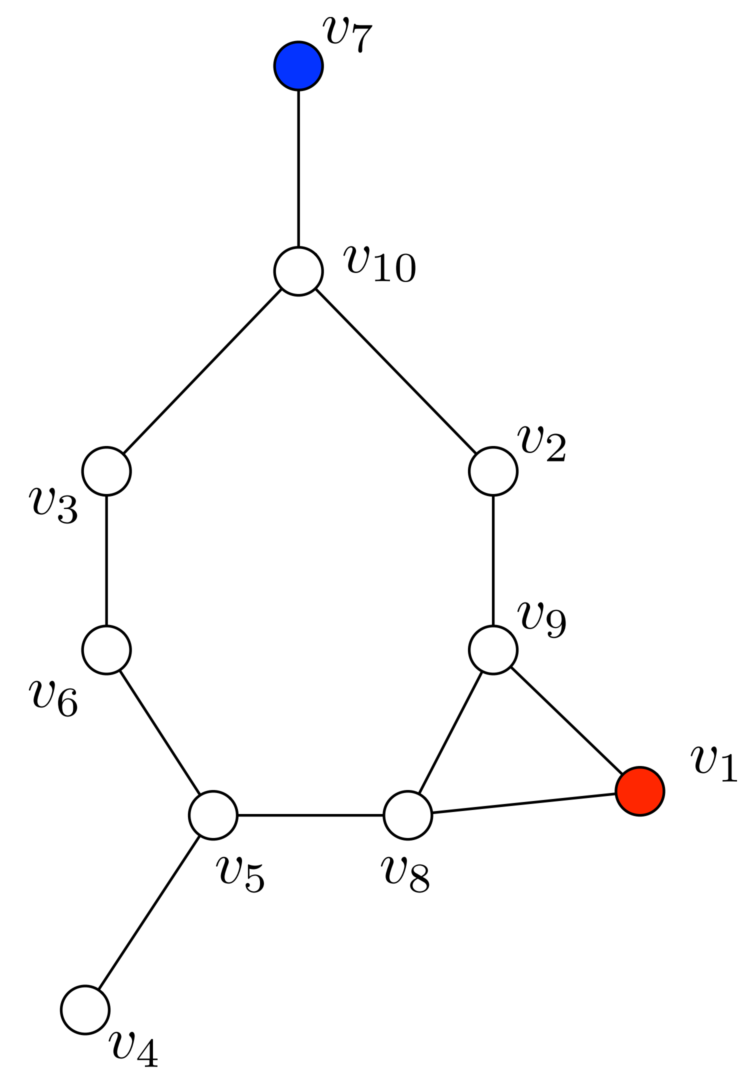


- the *path length* is the number of edges traversed
- there can be multiple (or no!) paths between two nodes
    - *Shortest Path*

Graph Theory

Example: *Shortest Path Problem*

Given a graph with two nodes identified as the ‘start’ node and the ‘terminal’ node, find the shortest length path between them



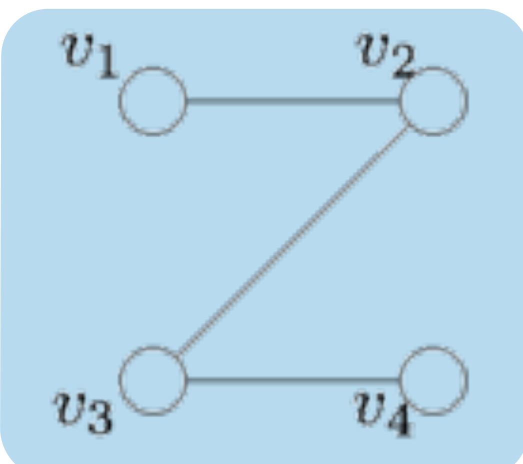
Dijkstra’s algorithm

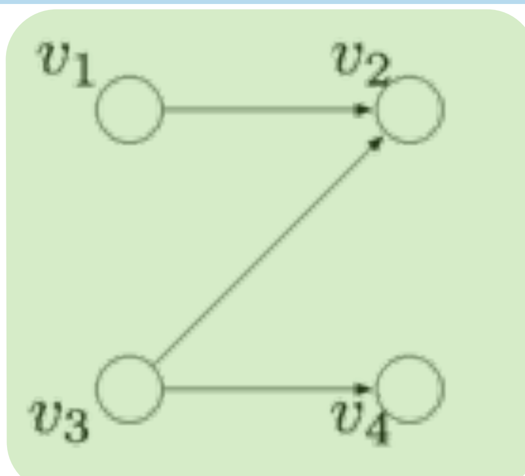
Graph Theory

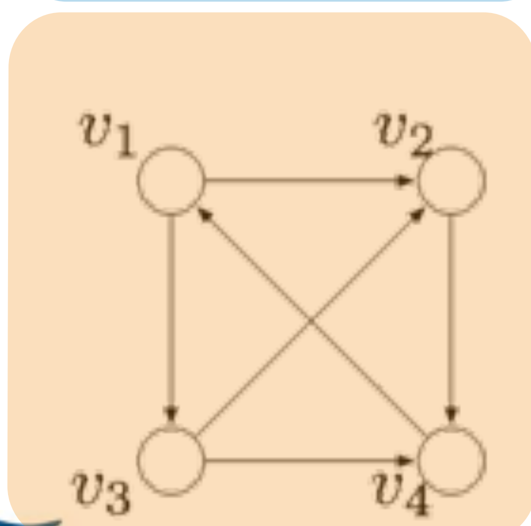
Undirected Graphs

connected

for every pair of vertices, there exists a path connecting them







disconnected

Directed Graphs

strongly connected

for every pair of vertices, there exists a *directed* path connecting them

weakly connected

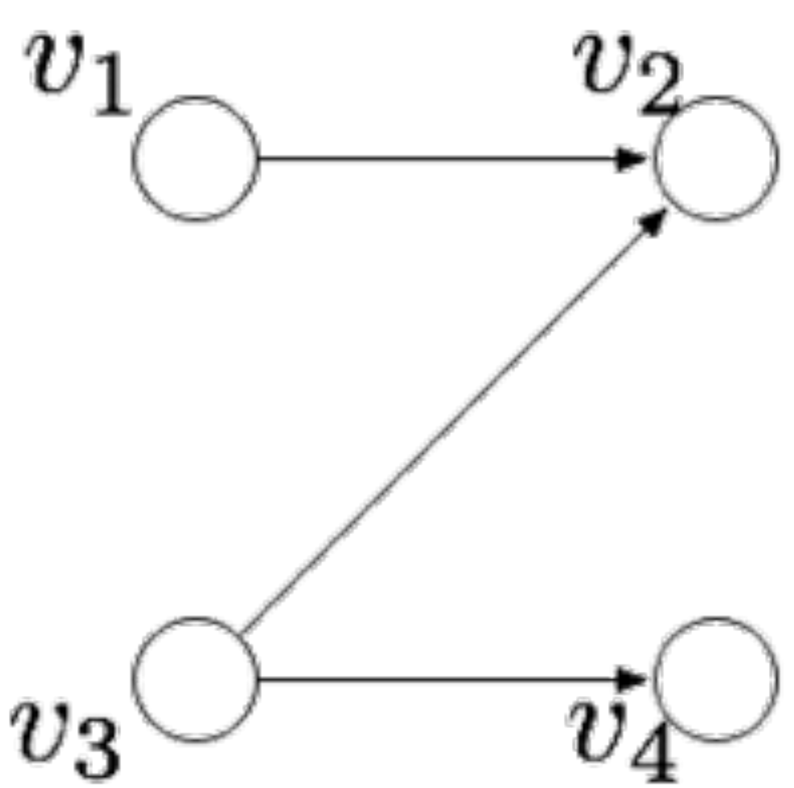
if the graph obtained by replacing each directed edge with an undirected edge is connected

Graph Theory

Undirected Graphs

Node Degree

$$d_i = |\mathcal{N}(v_i)|$$



Directed Graphs

In-Node Degree

Number of edges *entering* a node

Out-Node Degree

Number of edges *leaving* a node

$$d_1^{in} = 0$$

$$d_2^{in} = 2$$

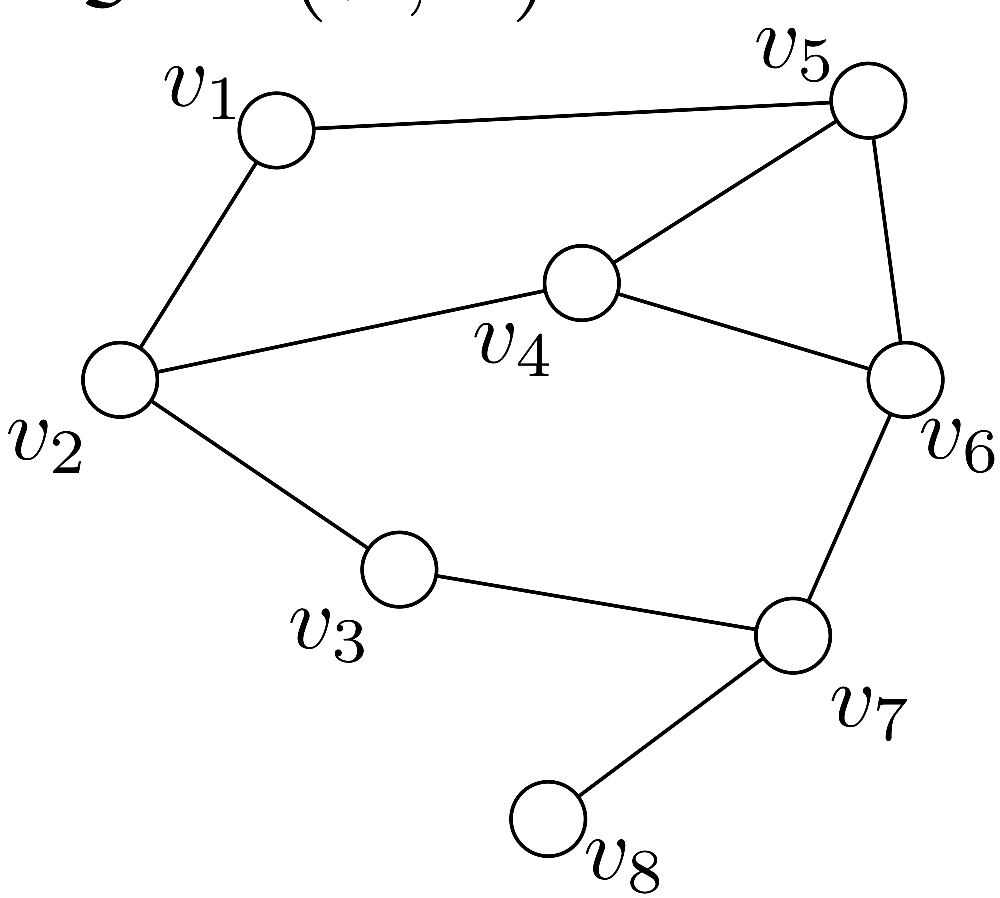
$$d_3^{out} = 2$$

Graph Theory

Graphs are a *set-theoretic* object!

Subgraphs

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

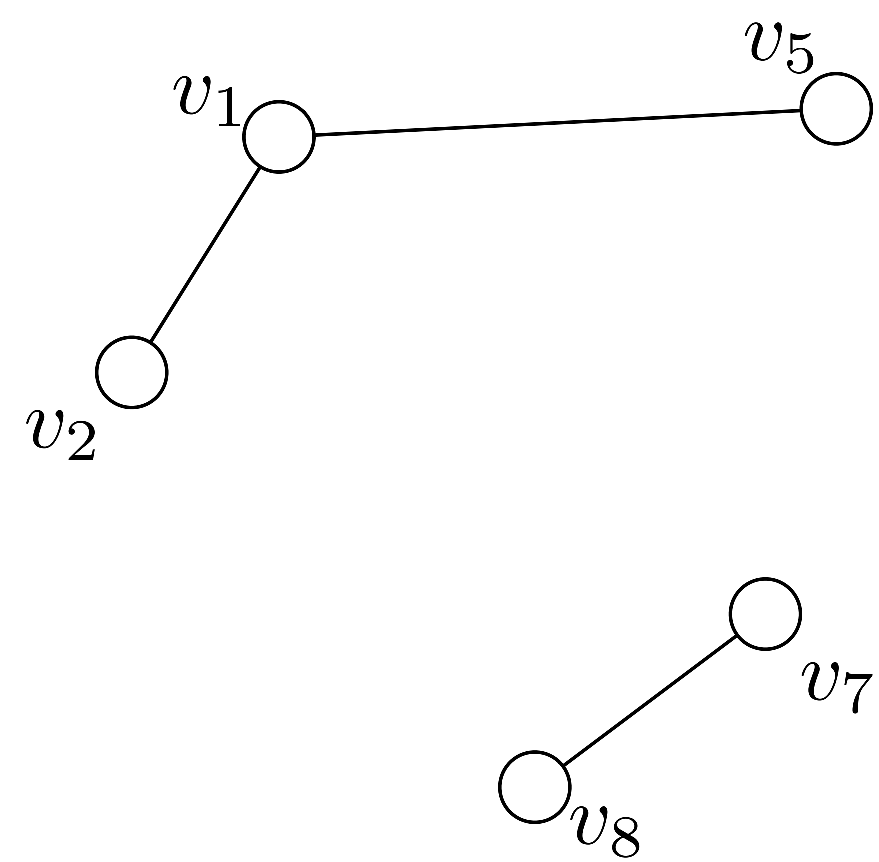


$\mathcal{V} = \{v_1, \ldots, v_8\}$

$\mathcal{V}' = \{v_1, v_2, v_5, v_8, v_7\} \subset \mathcal{V}$

$\mathcal{E}' \subset \mathcal{E}$

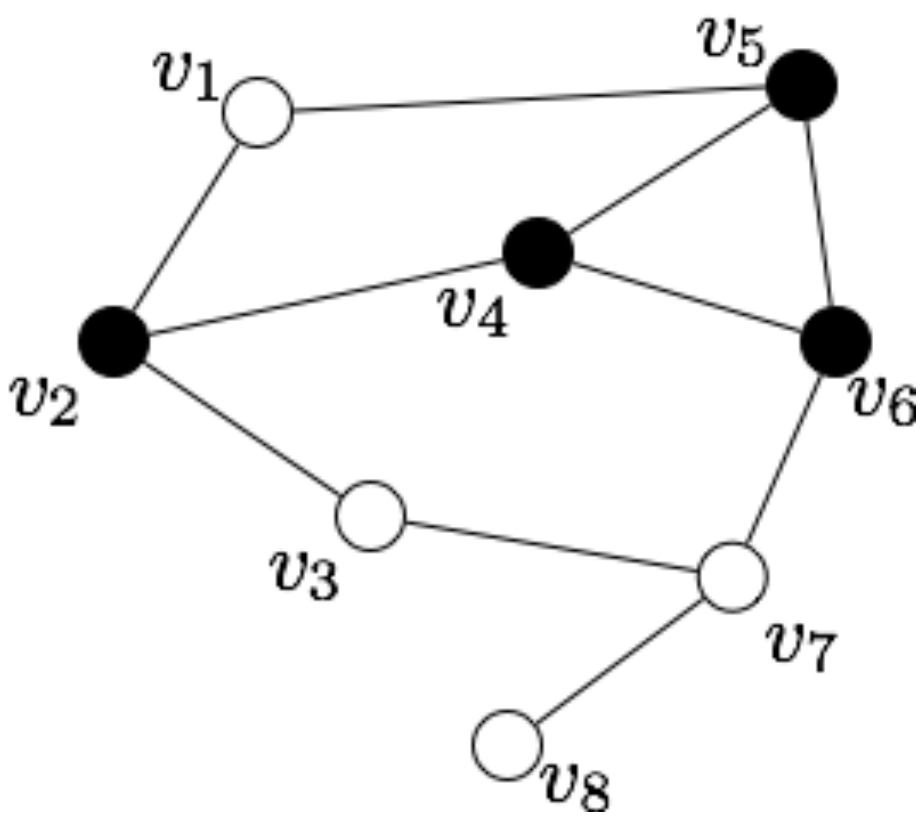
$\mathcal{G}' = (\mathcal{V}', \mathcal{E}') \subset \mathcal{G}$



Graphs are a *set-theoretic* object!

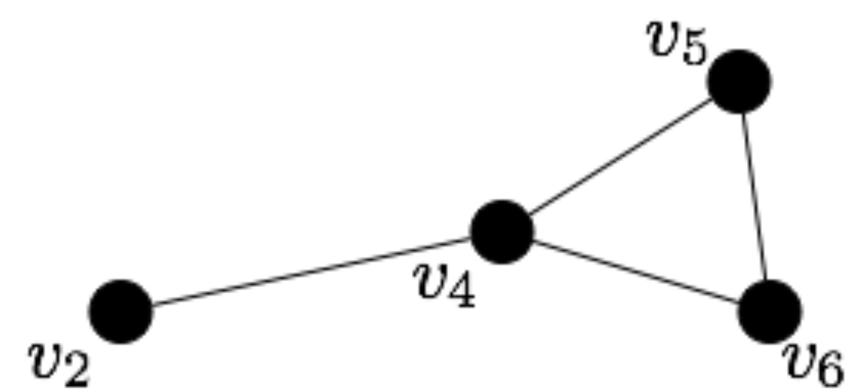
Induced Subgraphs

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



$$\mathcal{V} = \{v_1, \ldots, v_8\}$$

$$\mathcal{G}_S = (S, \mathcal{E}_S) \subseteq \mathcal{G}$$



$$S = \{v_2, v_4, v_5, v_6\}$$

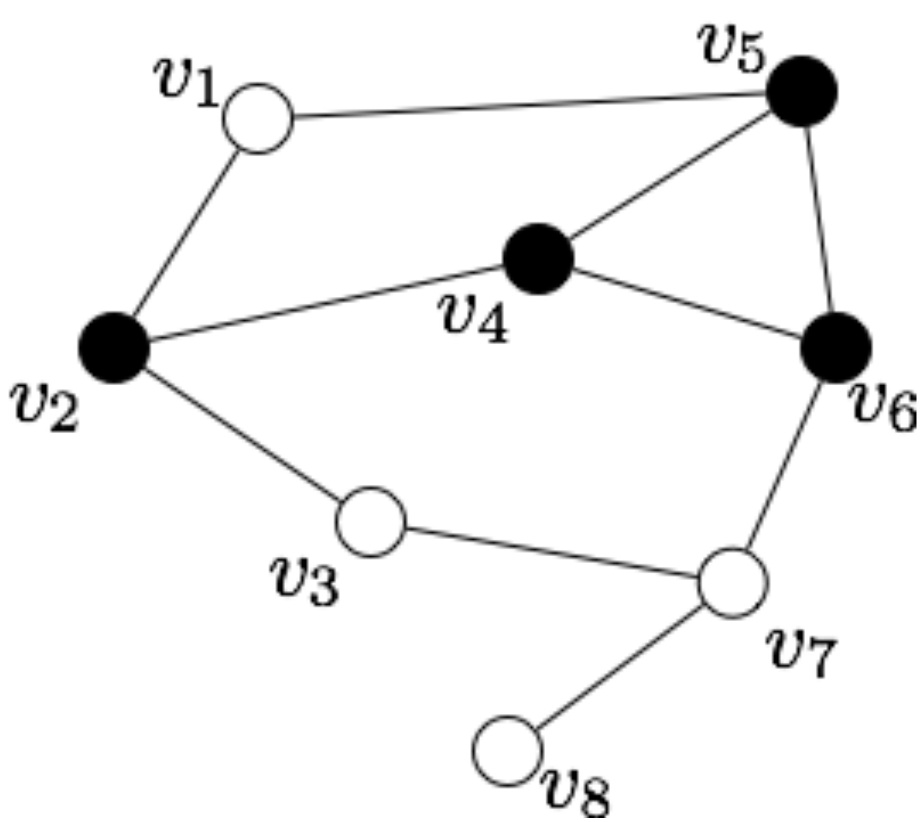
$$\mathcal{E}_S = \{\{v_i, v_j\} \in \mathcal{E} | v_i, v_j \in S\}$$

Graph Theory

Graphs are a *set-theoretic* object!

Induced Subgraphs

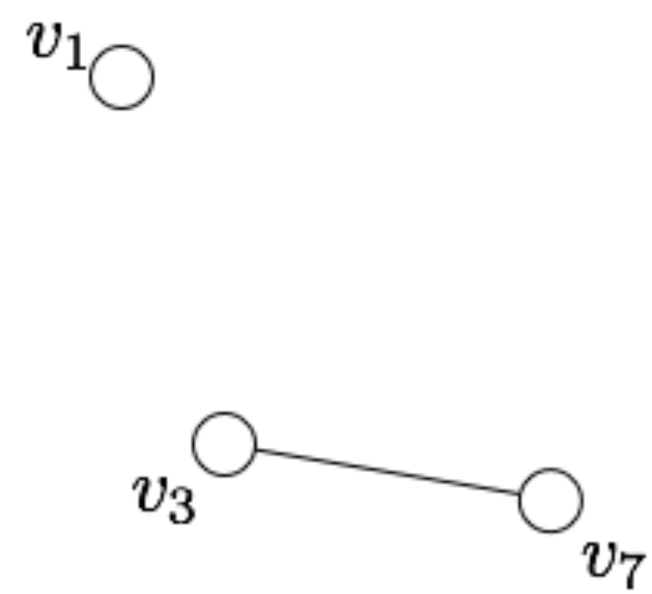
$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



$$\mathcal{V} = \{v_1, \ldots, v_8\}$$

boundary

$$\partial\mathcal{G}_S = (\partial S, \mathcal{E}_{\partial S})$$



$$\begin{aligned} \partial S &= \{v_i \in \mathcal{V} | v_i \notin S, \exists v_j \in S \; s.t. \; \{v_i, v_j\} \in \mathcal{E}\} \\ &= \{v_1, v_3, v_7\} \end{aligned}$$

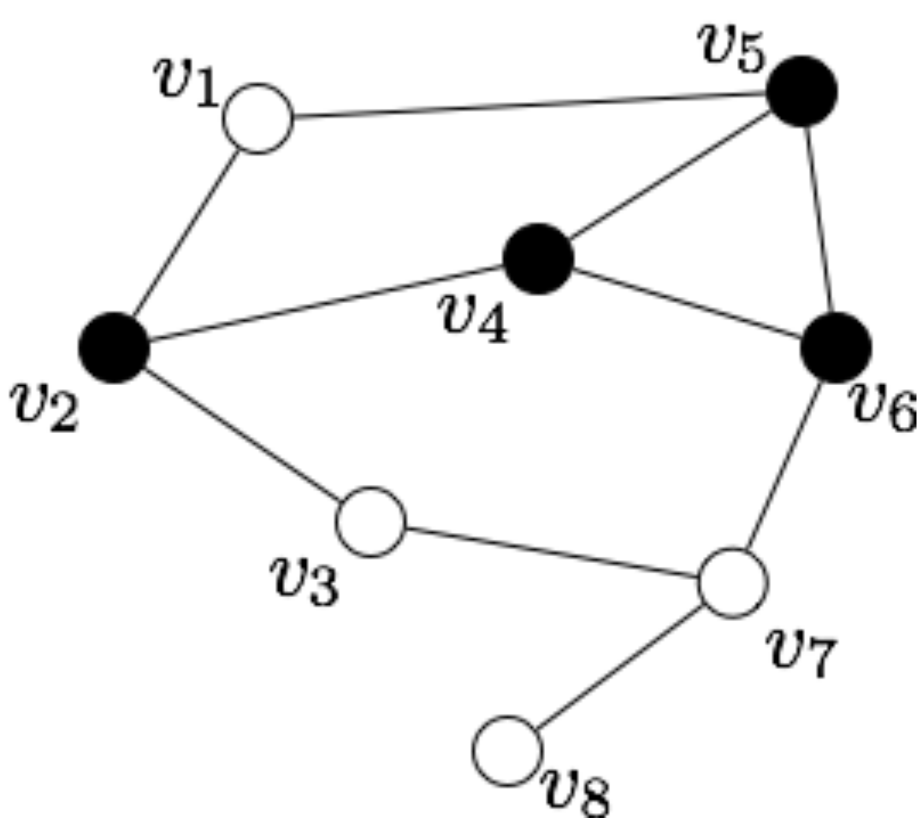
$$\mathcal{E}_{\partial S} = \{\{v_i, v_j\} \in \mathcal{E} \,|\, v_i, v_j \in \partial S\}$$

Graph Theory

Graphs are a *set-theoretic* object!

Induced Subgraphs

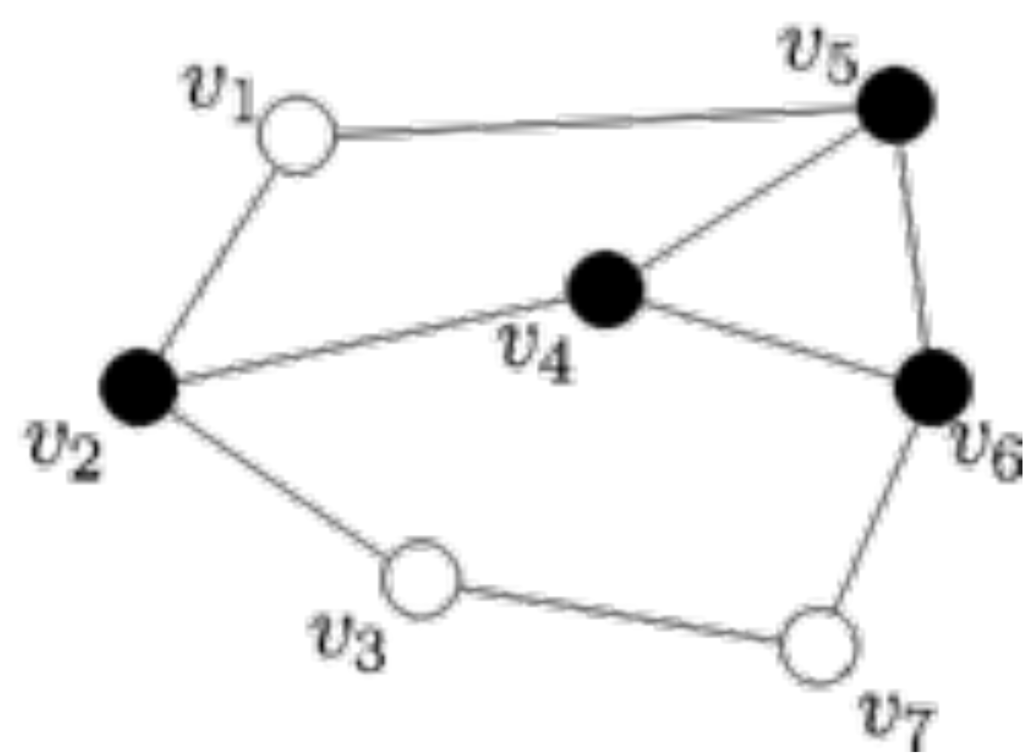
$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



$$\mathcal{V} = \{v_1, \ldots, v_8\}$$

closure

$$\mathbf{cl}\, \mathcal{G}_S = \mathcal{G}_S \cup \partial \mathcal{G}_S$$

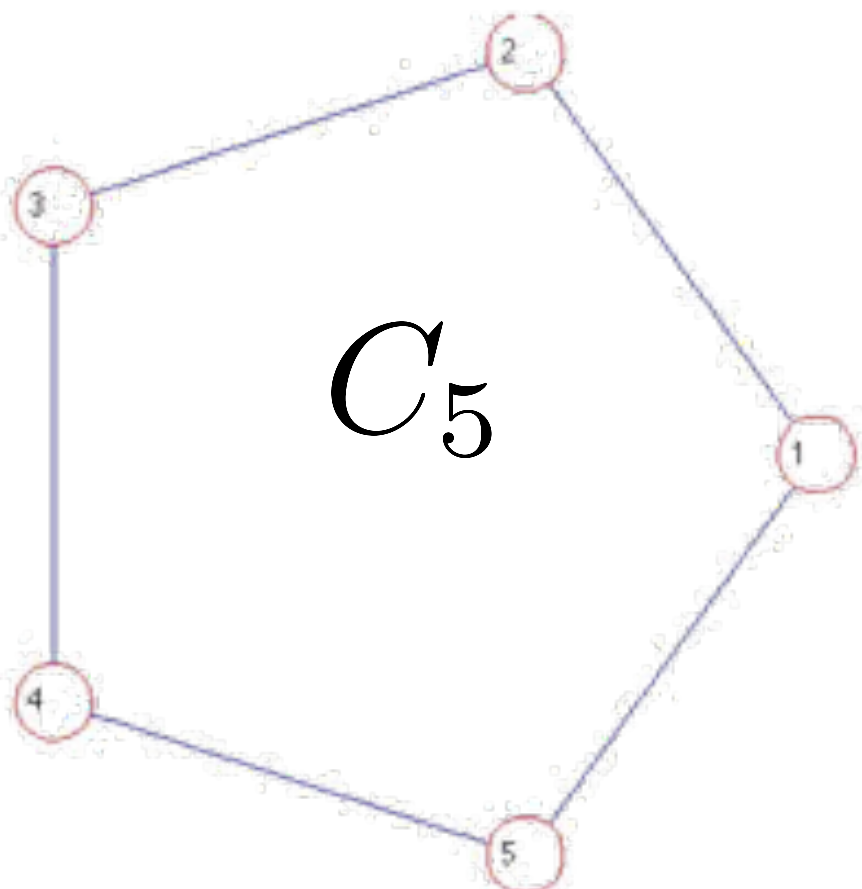


Graph Theory

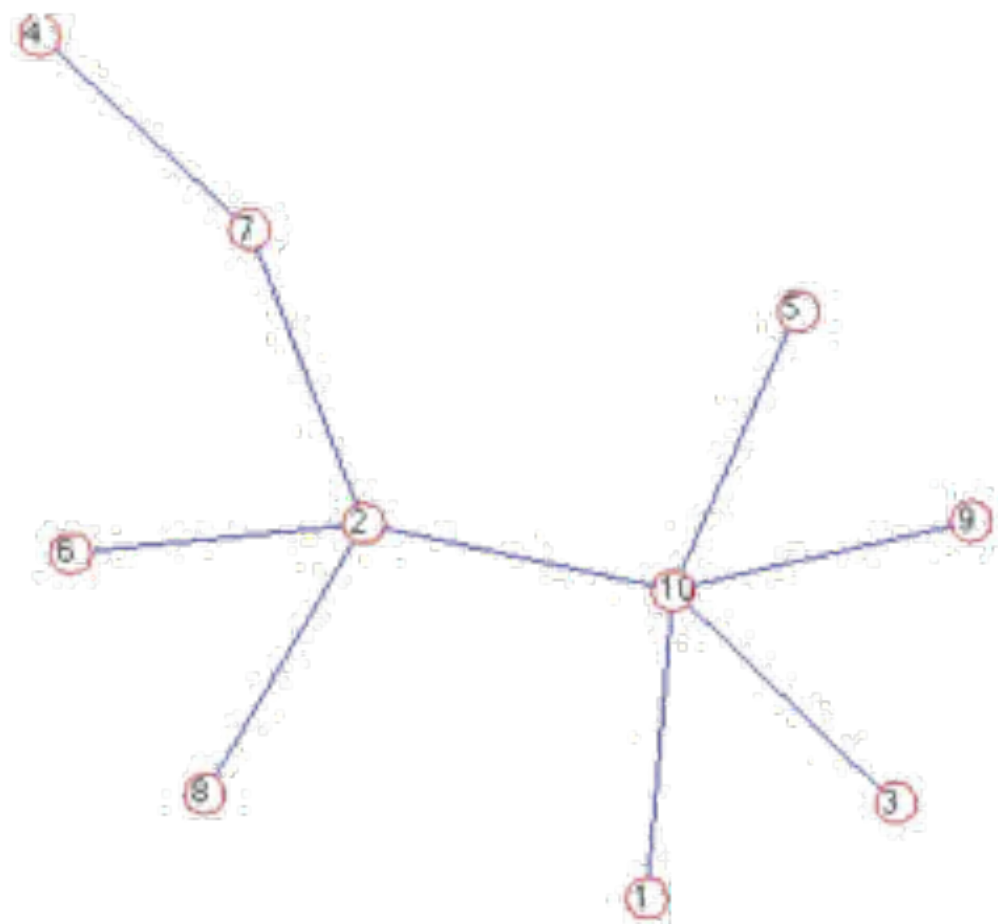
some special graphs…

Trees and Cycles

A *cycle* is a connected graph where each node has degree 2



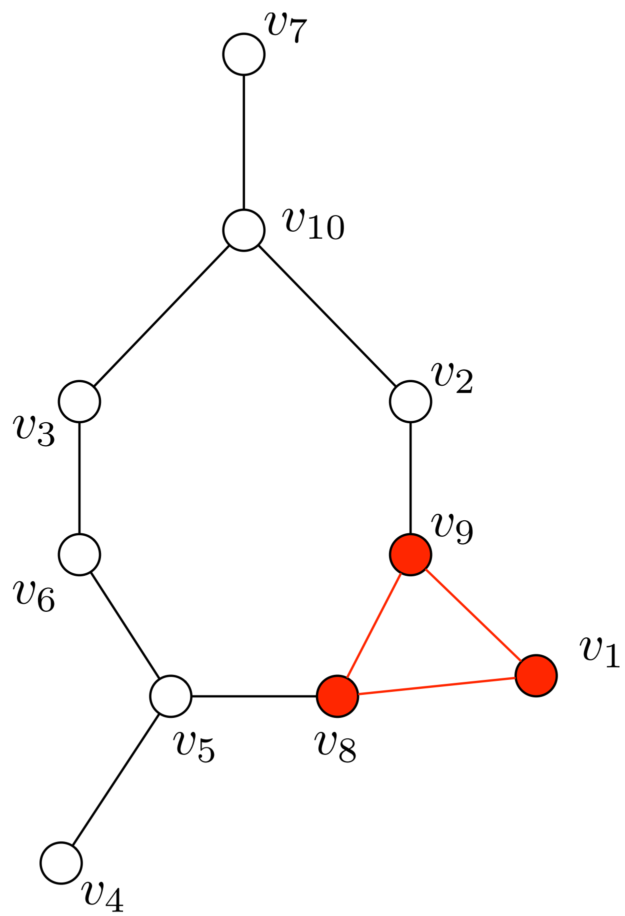
A *tree* is a connected graph containing no cycles



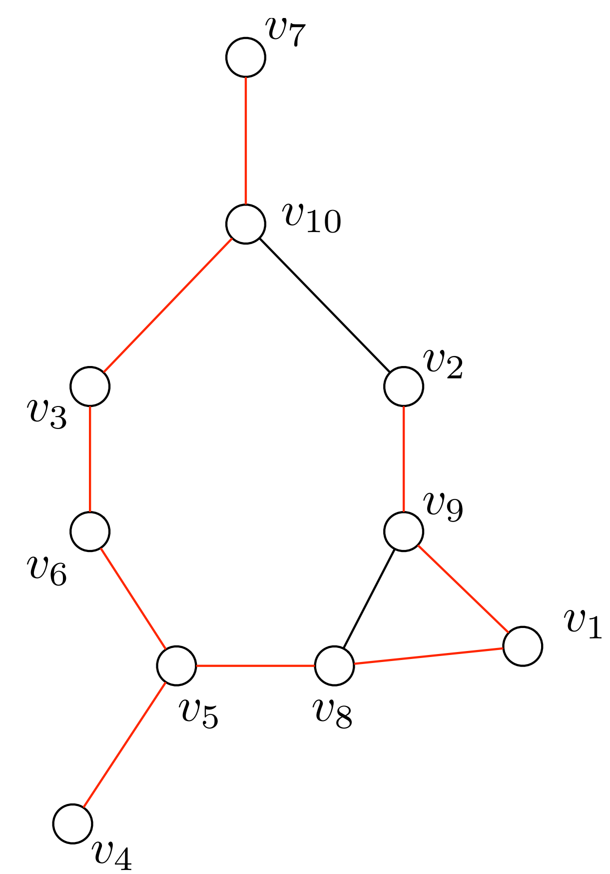
some special graphs…

Trees and Cycles

A graph contains *cycles* if there is a subgraph that is a cycle



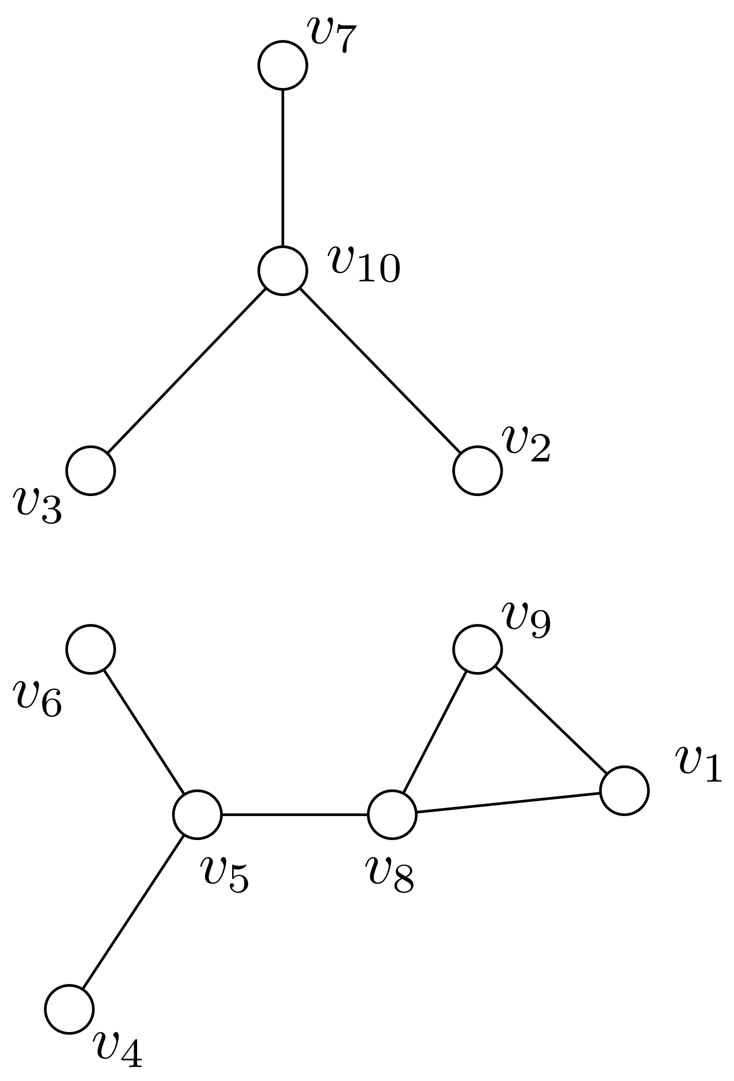
A *spanning tree* of a connected graph is a subgraph that is a tree



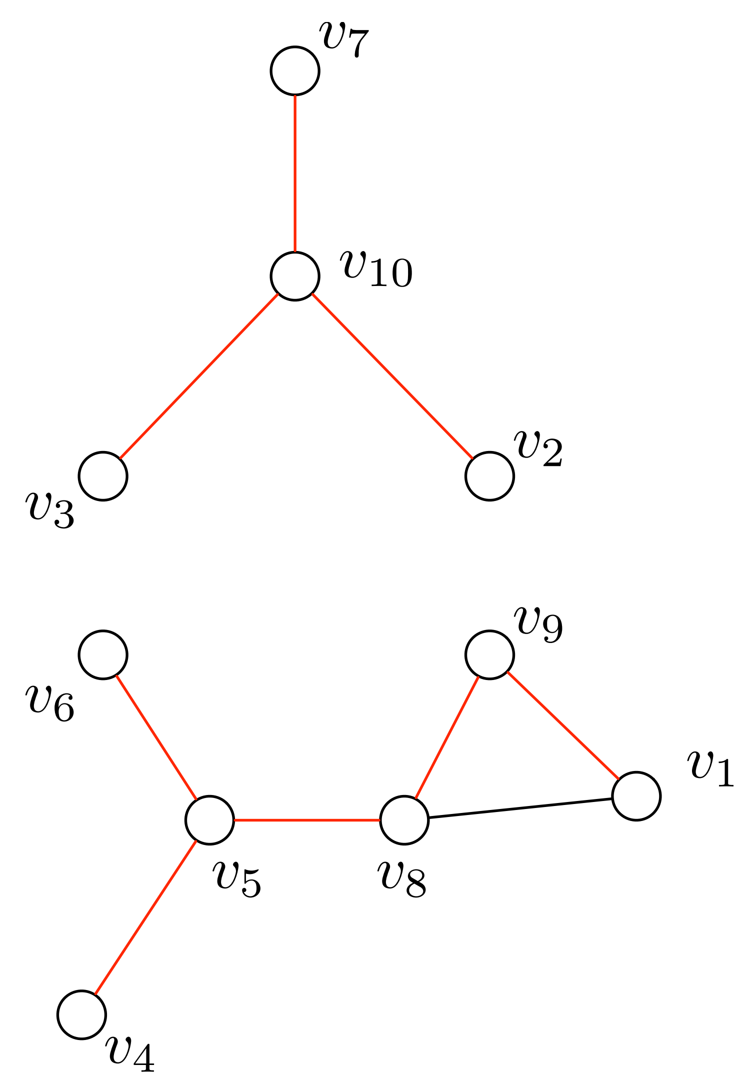
Graph Theory

some special graphs…

Forests



A *spanning forest* is a maximal acyclic subgraph

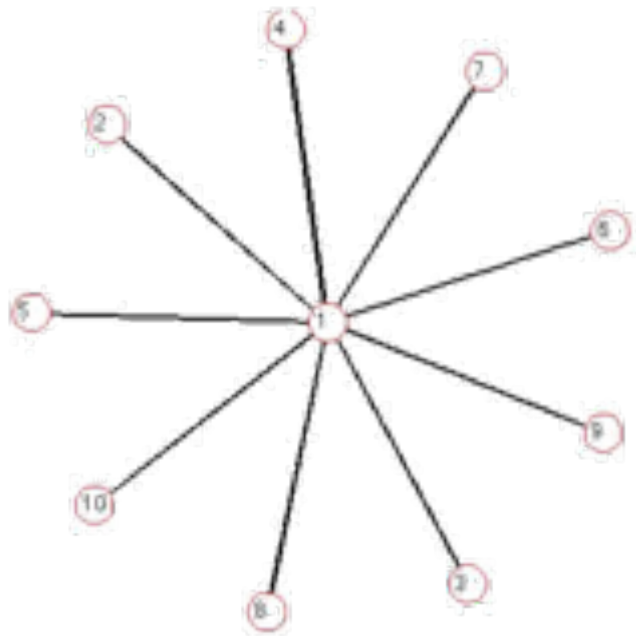


Graph Theory

some special graphs…

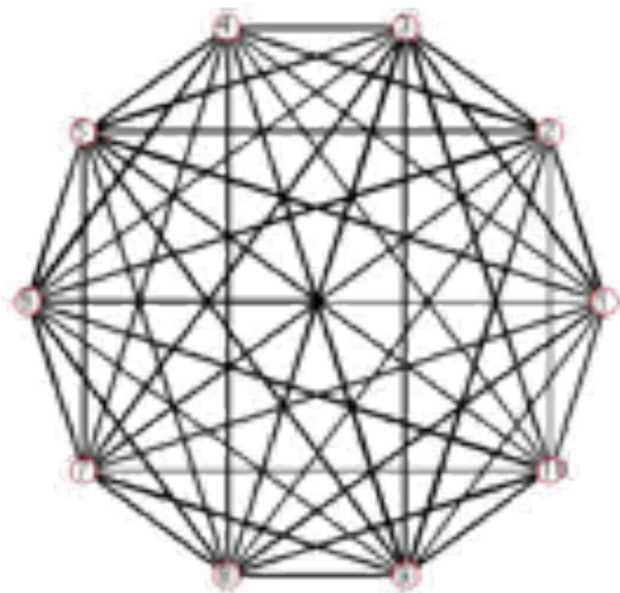
Star Graph

S_{10}



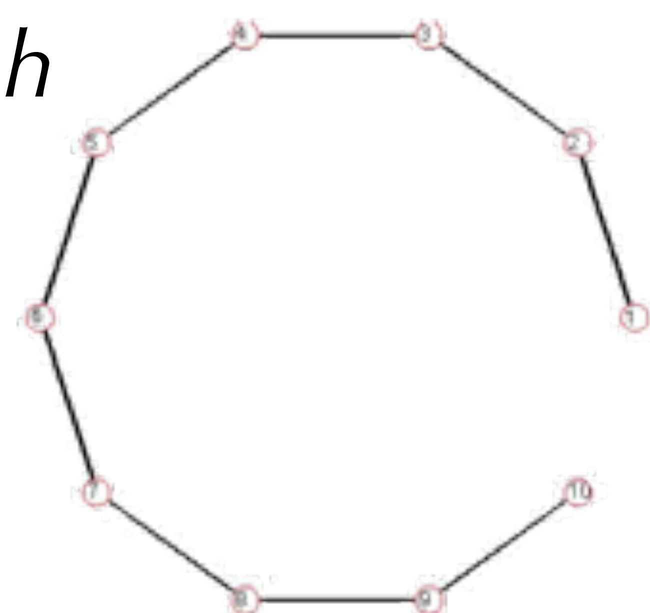
Complete Graph

K_{10}

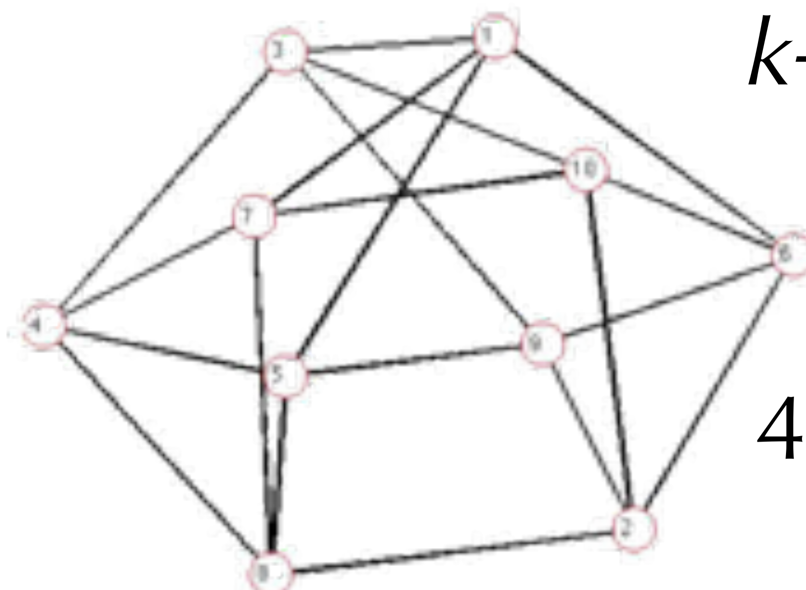


Path Graph

P_{10}



k-Regular Graph

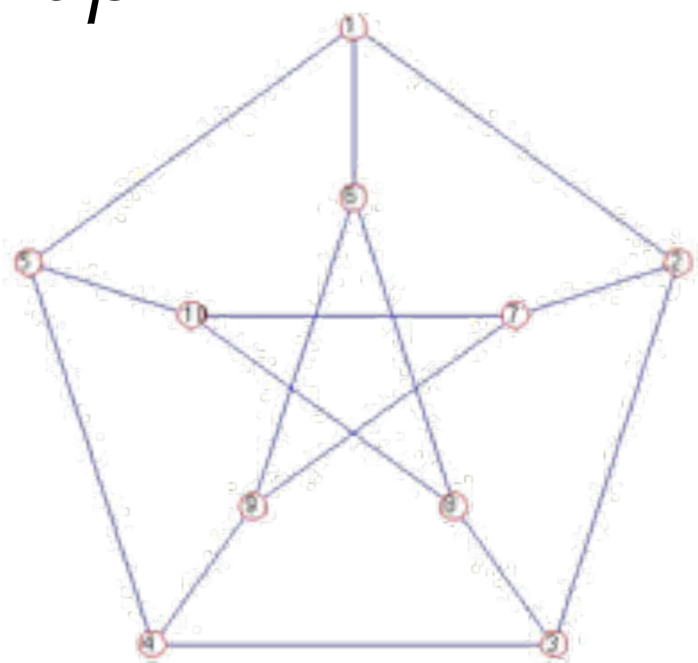


4-regular

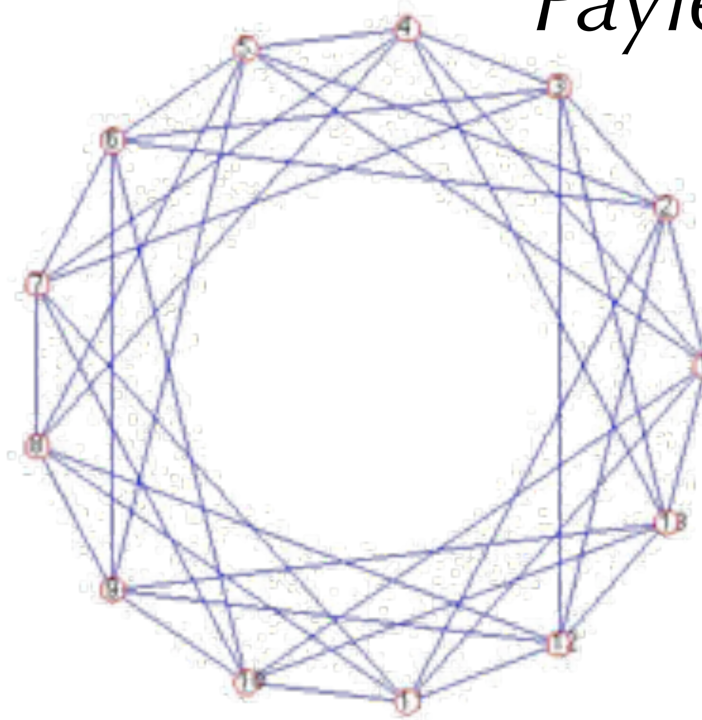
Graph Theory

some special graphs...

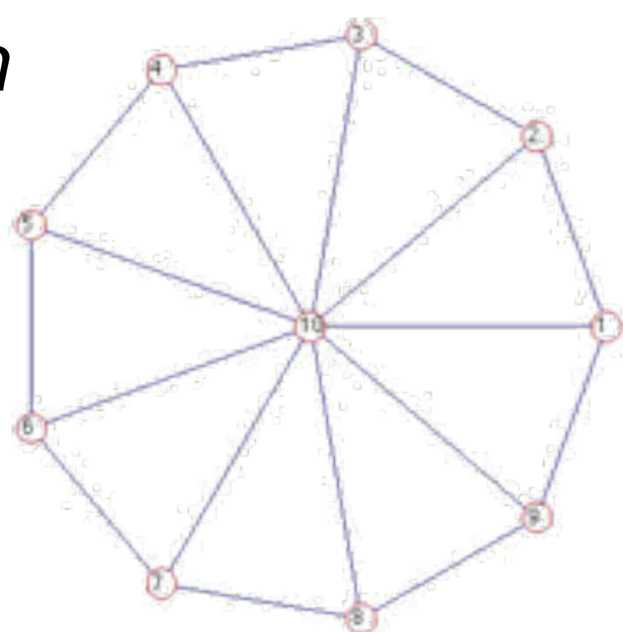
Peterson Graph



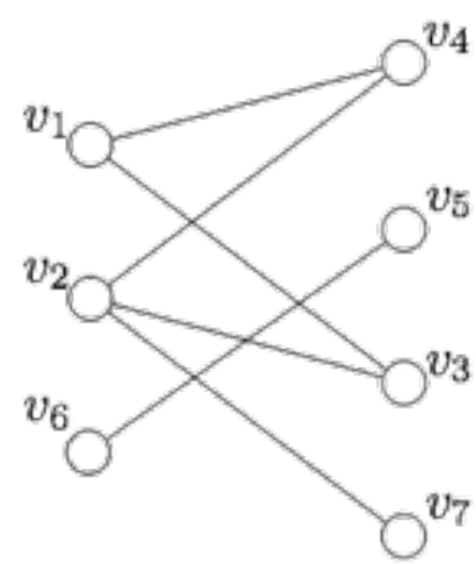
Payley Graph



Wheel Graph

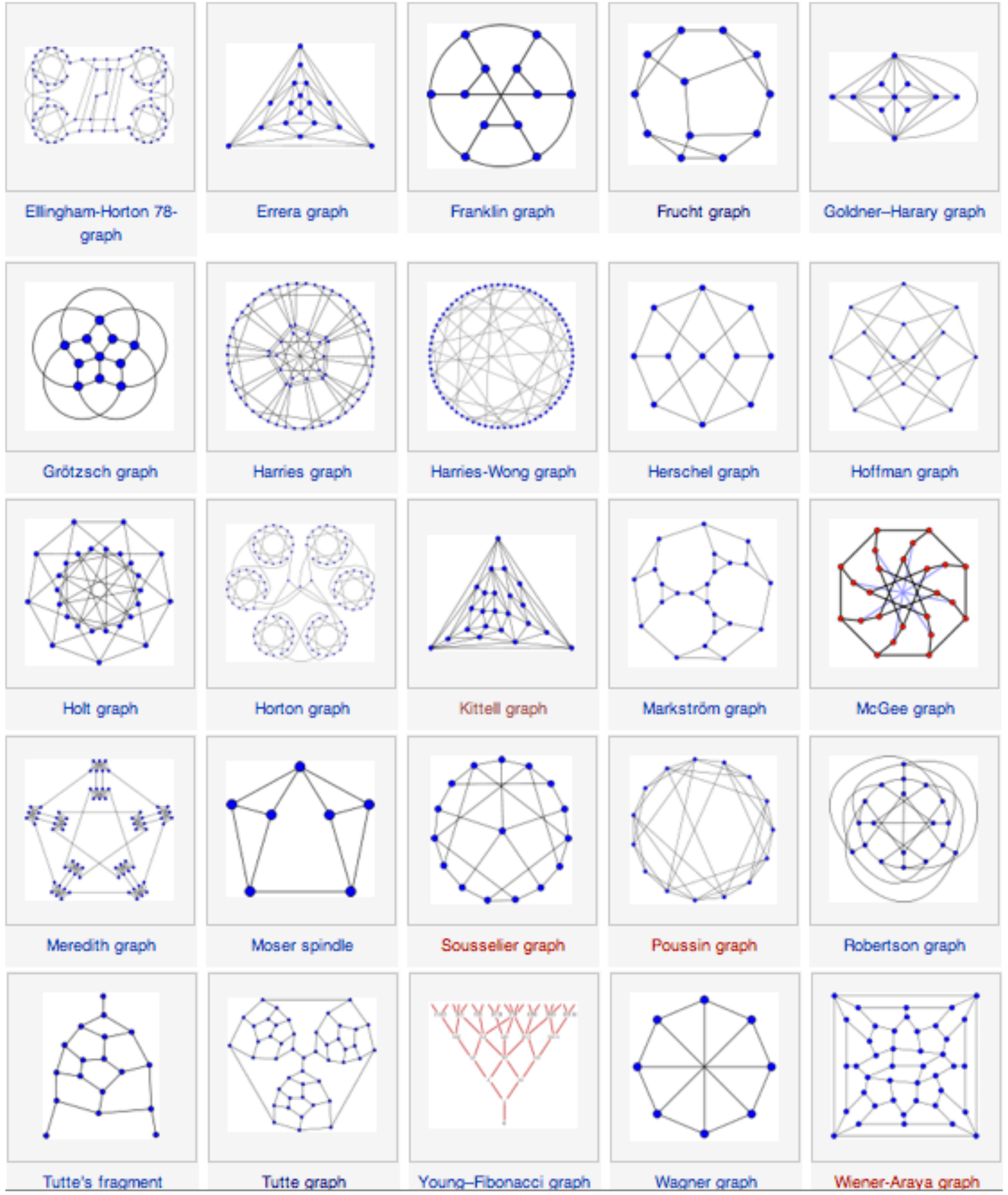


Bipartite Graph



Graph Theory

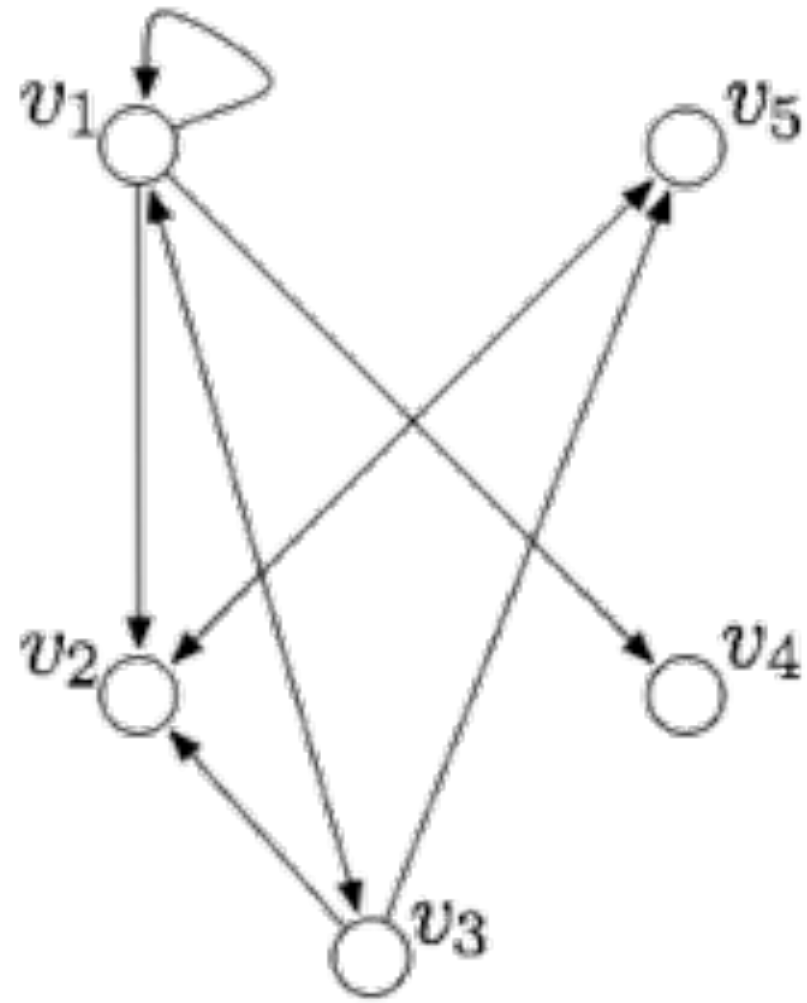
so many named graphs!



Graph Theory

Example: All square matrices have a graph representation

$$M = \begin{bmatrix} 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Longleftrightarrow$$



Graph of a Matrix $M \in \mathbb{R}^{n \times n}$

$$\mathcal{G}(M) = (\mathcal{V}(M), \mathcal{E}(M))$$

$$|\mathcal{V}(M)| = n \qquad e = (v_i, v_j) \in \mathcal{E}(M) \Leftrightarrow [M]_{ij} \neq 0$$

Graph Theory

Definition

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *irreducible* if there does not exist a permutation matrix P and an integer r such that

$$P^T M P = \left[\begin{array}{cc} B & C \\ 0 & D \end{array}\right],$$

with $B \in \mathbb{R}^{r\times r}$, $C \in \mathbb{R}^{r\times n-r}$, and $D \in \mathbb{R}^{n-r\times n-r}$.

$$M = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$$

irreducible

$$A = \left[\begin{array}{cc} 0 & 0 \\ 1 & 2 \end{array}\right]$$

reducible $P = ?$

Theorem

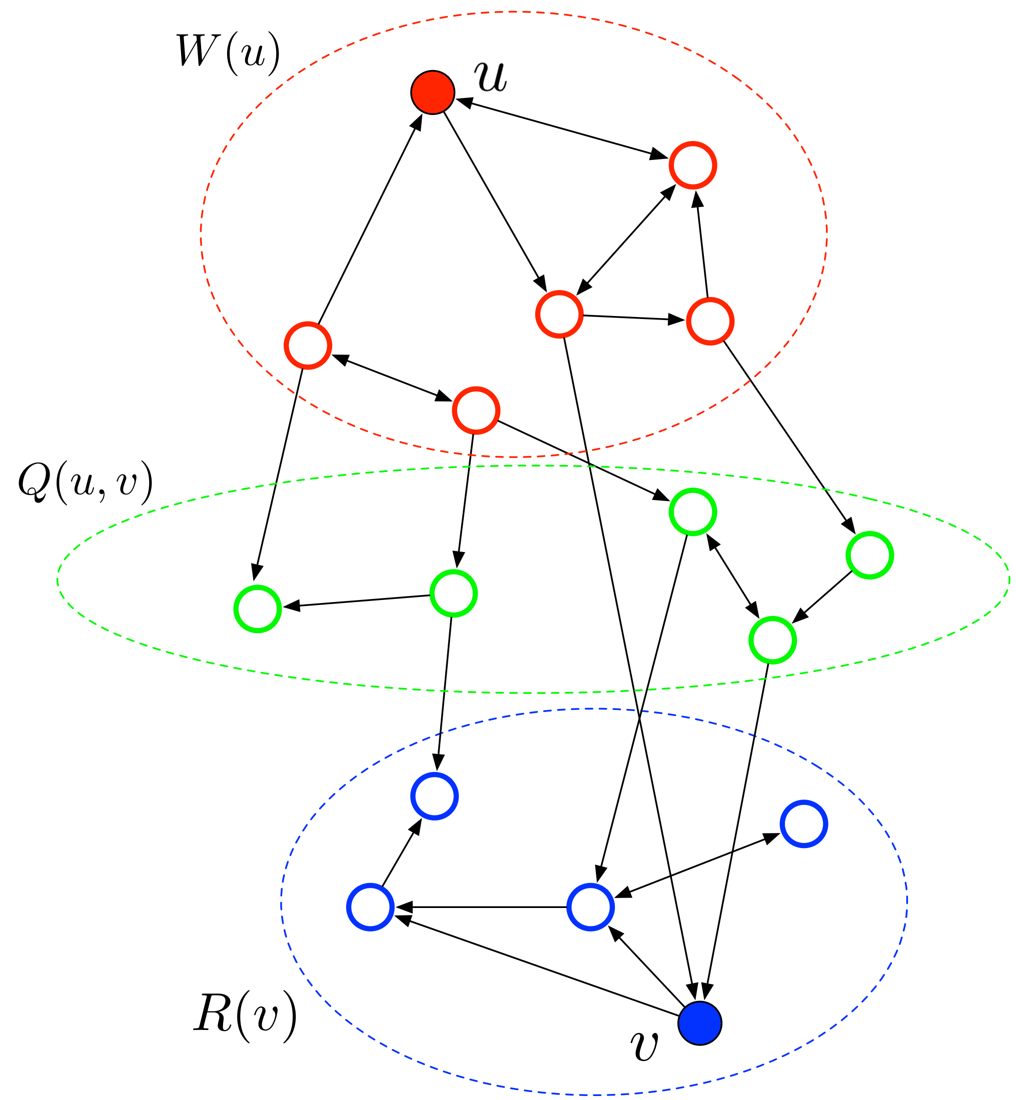
Let $M \in \mathbb{R}^{n\times n}$. The following are equivalent:

1. M is irreducible,
2. The digraph associated with M ($\mathcal{G}(M)$) is strongly connected.

Proof

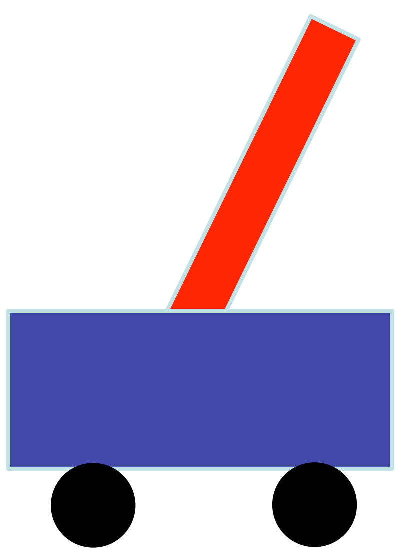
M is irreducible $\Rightarrow \mathcal{G}(M)$ is strongly connected

assume the graph is *not* strongly connected



Example: Structured Linear System

A *structured linear system* is a description of a dynamic system that considers only the interaction and influence between system states, control, and outputs independent of any realization of parameter values



$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_p g}{m_c} & \frac{-K_1^2}{R_m m_c} & 0 \\ 0 & \frac{(m_p+m_c)g}{m_c I_p} & \frac{K_1^2}{R_m m_c I_p} & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{R_m m_c} \\ \frac{-K_1}{R_m m_c I_p} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$

Example: Structured Linear System

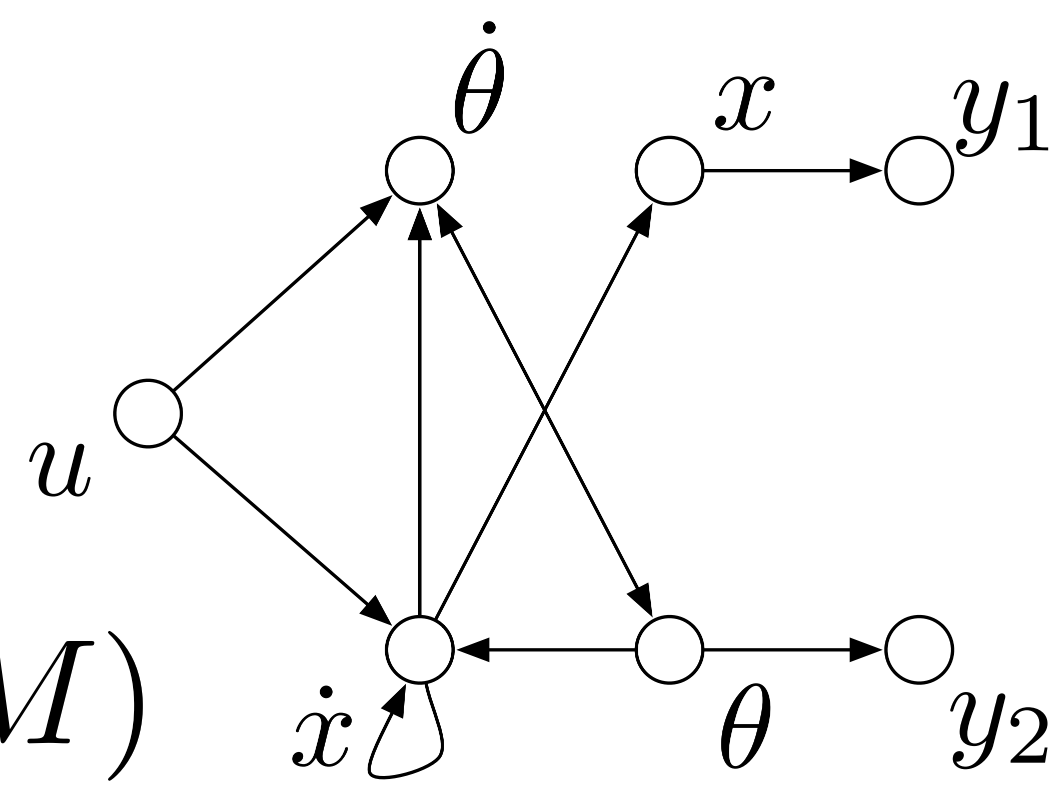
$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_p g}{m_c} & \frac{-K_1^2}{R_m m_c} & 0 \\ 0 & \frac{(m_p+m_c)g}{m_c I_p} & \frac{K_1^2}{R_m m_c I_p} & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{R_m m_c} \\ \frac{-K_1}{R_m m_c I_p} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

$$\mathcal{V} = \{u, x, \dot{x}, \theta, \dot{\theta}, y_1, y_2\}$$

$\mathcal{G}(M)$



Graph Theory

Example: Structured Linear System

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix} u$$

Definition

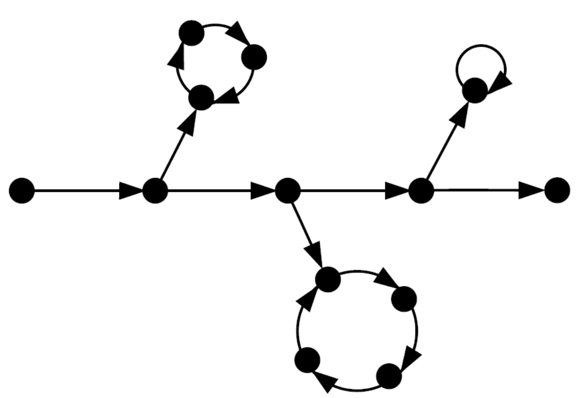
A system (A, B) is *structurally controllable* if there exists a system structurally equivalent to (A, B) which is controllable in the usual sense.

Example: Structured Linear System

Theorem [Lin '74]

The following statements for a structured system (A, B) are equivalent:

- (A, B) is structurally controllable
- In the graph $\mathcal{G}(A, B)$, there exists a disjoint union of cacti that covers all the state vertices.



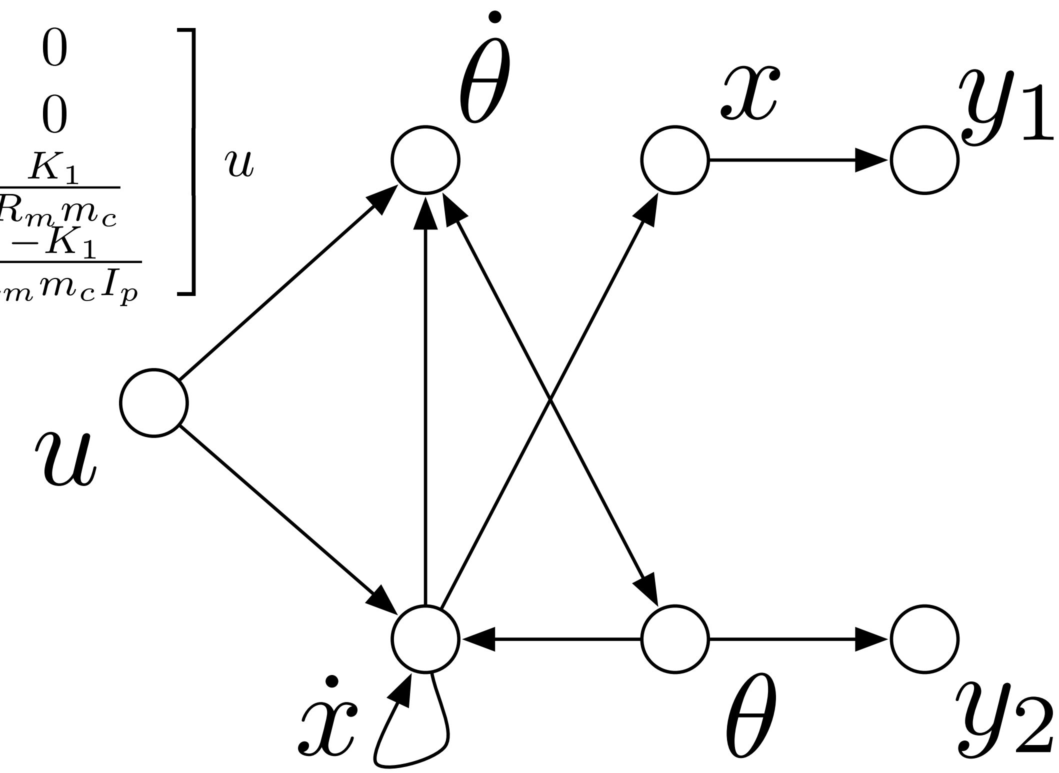
A "cactus" with three "buds"

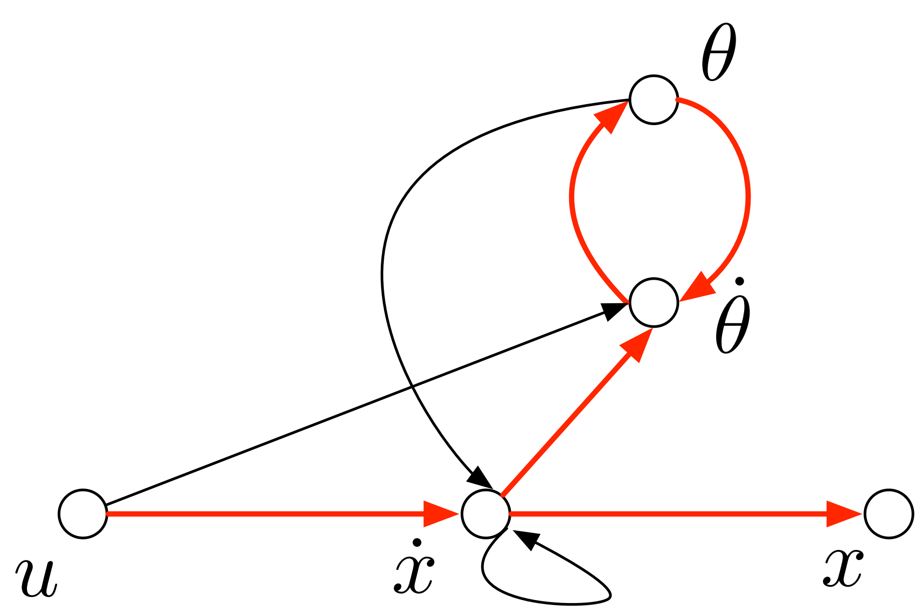
Graph Theory

Example: Structured Linear System

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_p g}{m_c} & \frac{-K_1^2}{R_m m_c} & 0 \\ 0 & \frac{(m_p+m_c)g}{m_c I_p} & \frac{K_1^2}{R_m m_c I_p} & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{R_m m_c} \\ \frac{-K_1}{R_m m_c I_p} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$

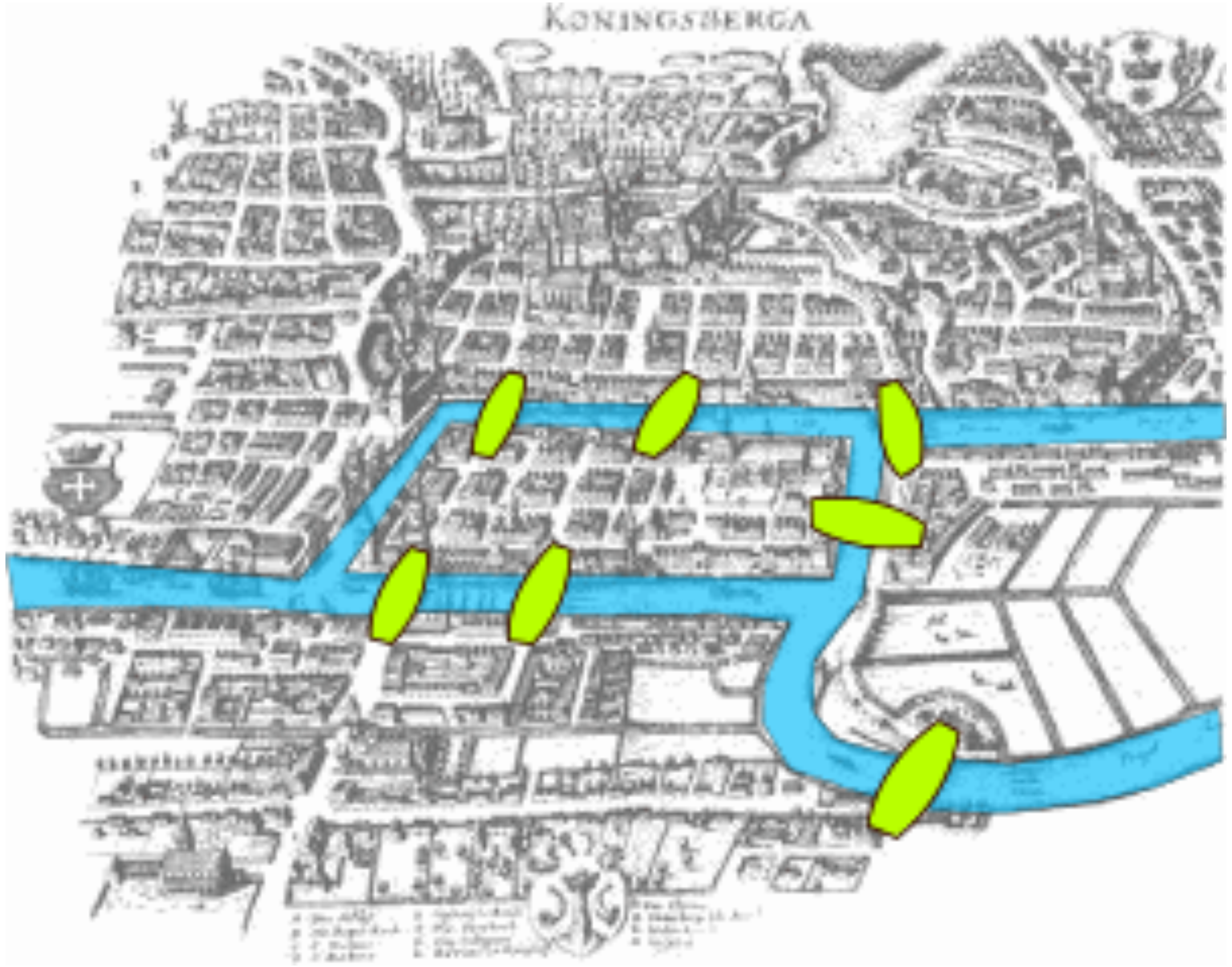




Graph Theory

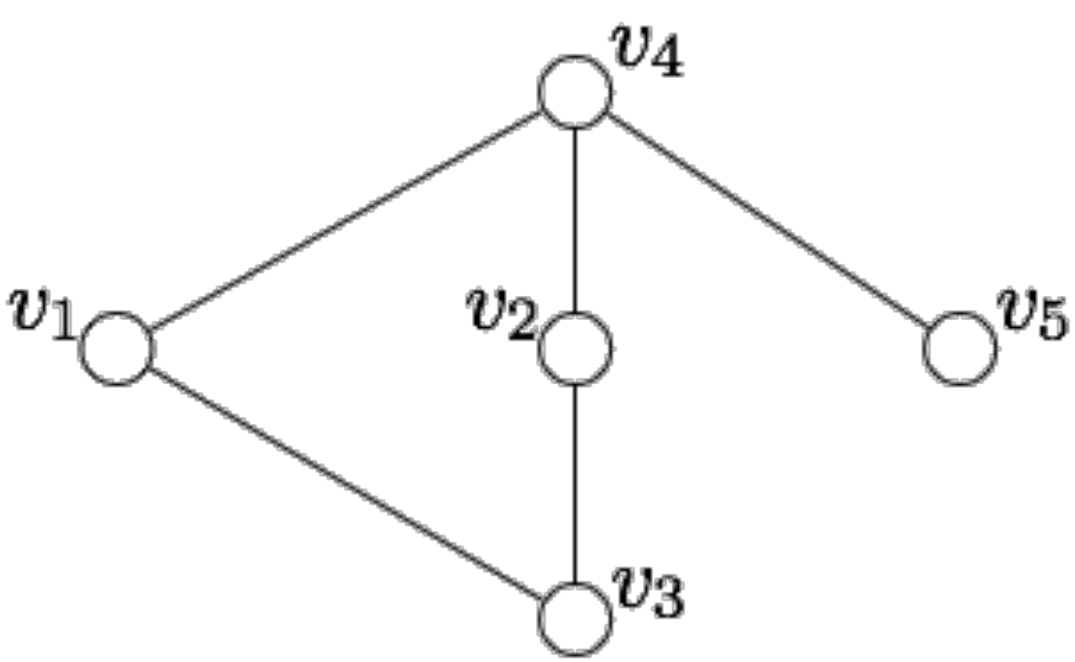
Example: Seven Bridges of Königsberg (Euler 1735)

Is there a *walk* through the city of Königsberg that crosses each bridge once and *only* once?



Algebraic Graph Theory

Graphs can be described using matrices



Degree Matrix

$$\Delta(\mathcal{G}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonal matrix with degree of each node on diagonal

Adjacency Matrix

$$A(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

symmetric matrix encoding adjacency relationship of nodes in graph

Lemma

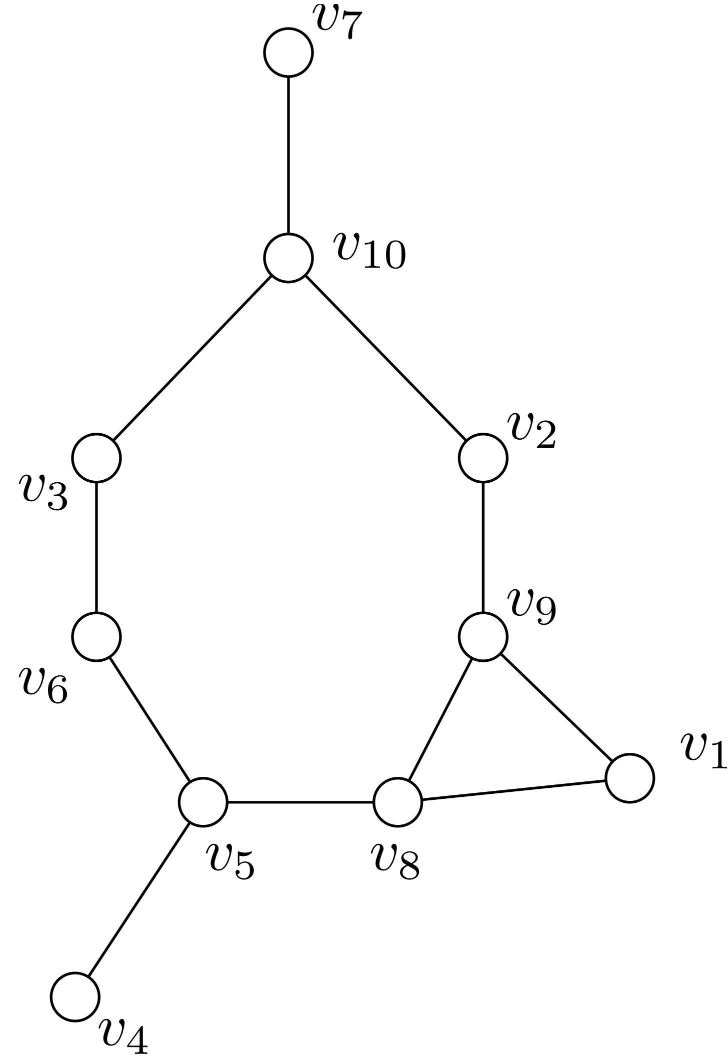
Let $\mathcal{G}$ be a graph with adjacency matrix $A(\mathcal{G})$. The number of walks from node v_i to v_j of length r is $[A(\mathcal{G})^r]_{ij}$

Algebraic Graph Theory

Corollary

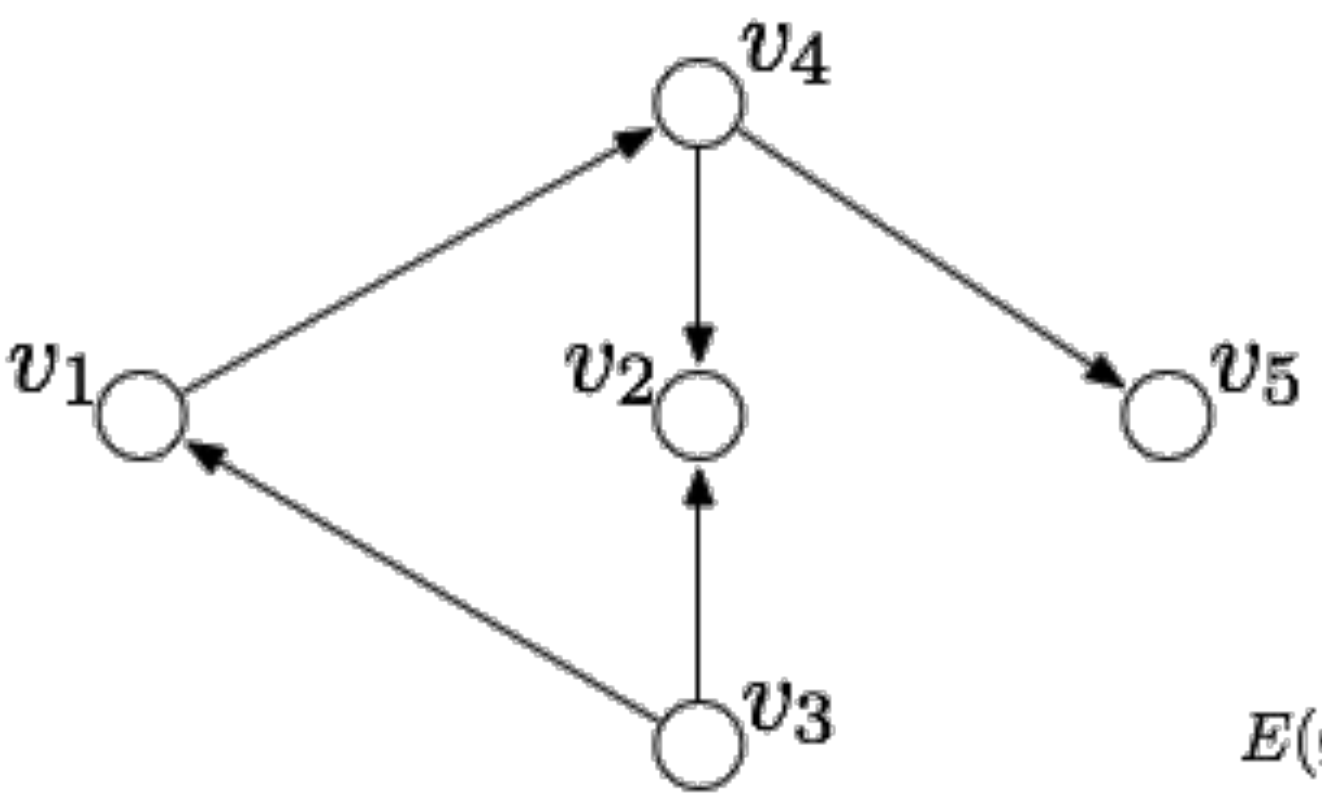
Let $\mathcal{G}$ be a graph with e edges, t triangles, and adjacency matrix $A(\mathcal{G})$. Then

- $\text{trace}\, A(\mathcal{G}) = 0$
- $\text{trace}\, A(\mathcal{G})^2 = 2e$
- $\text{trace}\, A(\mathcal{G})^3 = 6t$



Algebraic Graph Theory

Graphs can be described using matrices



assign arbitrary orientation
to each edge

Incidence Matrix

$$E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$$

$$E(\mathcal{G}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Theorem

Let $\mathcal{G}$ be a graph with n vertices, c connected components, and an arbitrary orientation assigned to each edge. Then $\operatorname{rank} E(\mathcal{G}) = n - c$.

Proof

Suppose there exists an $x \in \mathbb{R}^n$ such that $x^T E(\mathcal{G}) = 0$. If $(u, v) \in \mathcal{E}(\mathcal{G})$, this implies that $x_u - x_v = 0$. If we consider x as a function on the nodes of the graph, then it must be constant on any connected component of $\mathcal{G}$. By assumption, there are c such components.

Algebraic Graph Theory

Example: Relative Sensing Networks

Interferometry is a technique used for imaging in deep space. Rather than using 1 large (and expensive!) telescope, a team of smaller (and cheaper!) sensors can achieve the same goal. This requires high accuracy and precision of *relative spacing* between satellites.

$$\dot{x}_i(t) = f(x_i(t), u_i(t), t)$$

$$e_k = \{v_i, v_j\} \in \mathcal{E}$$

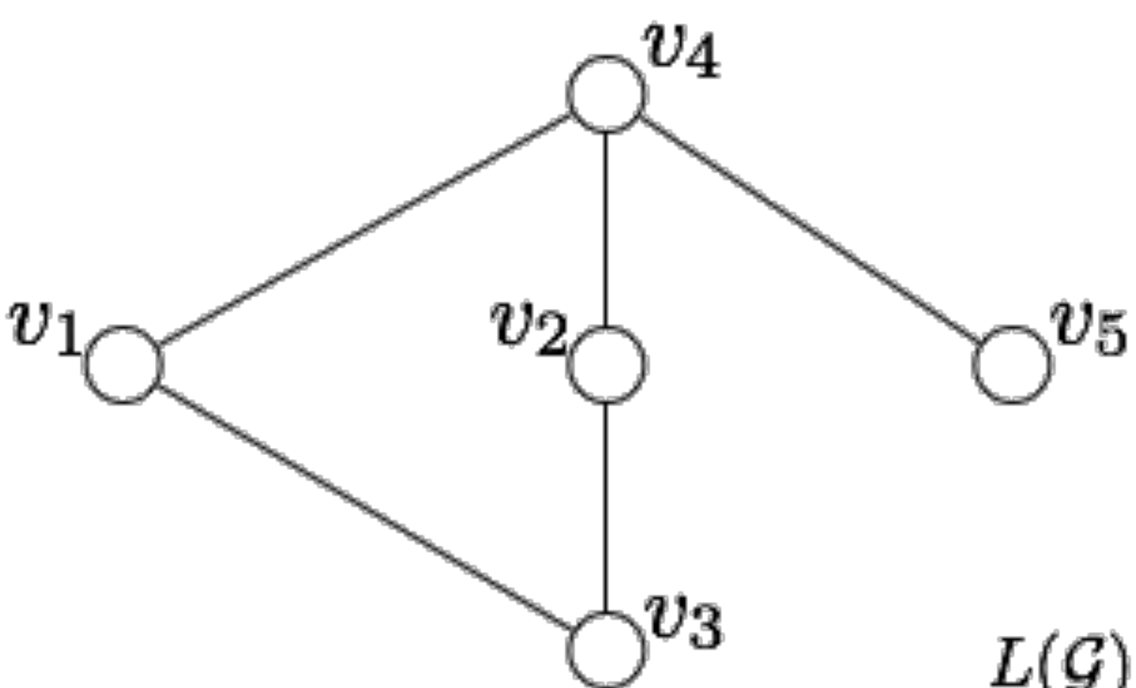
$$y_k(t) = x_i(t) - x_j(t)$$

$$\mathbf{y}(t) = E(\mathcal{G})^T \mathbf{x}(t)$$

Algebraic Graph Theory

The Combinatorial (graph) Laplacian Matrix

$$\begin{aligned} L(\mathcal{G}) &= \Delta(\mathcal{G}) - A(\mathcal{G}) \\ &= E(\mathcal{G})E(\mathcal{G})^T \end{aligned}$$



$$L(\mathcal{G}) = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Algebraic Graph Theory

The Combinatorial (graph) Laplacian Matrix and *Spectral Graph Theory*

$$L(\mathcal{G}) \in \mathbb{R}^{n \times n}$$

for a connected graph, there is a single eigenvalue at the origin

$$L(\mathcal{G})\mathbf{1} = 0$$

$$0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \ldots \leq \lambda_n(\mathcal{G})$$

algebraic connectivity of graph
Fiedler Eigenvalue

$$\lambda_2(\mathcal{G})$$

$$\text{trace}\, L(\mathcal{G}) = 2|\mathcal{E}|$$

Algebraic Graph Theory

Theorem

The graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$.

Theorem (Matrix Tree Theorem)

Let $\tau(\mathcal{G})$ be the number of spanning trees in $\mathcal{G}$. Then

$$\tau(\mathcal{G}) = \det\, L(\mathcal{G})_{(ij)}.$$