Analysis and Control of Multi-Agent Systems

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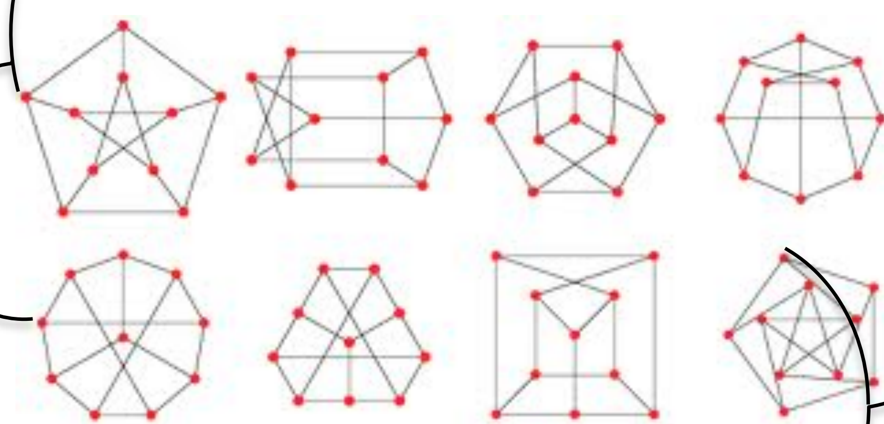
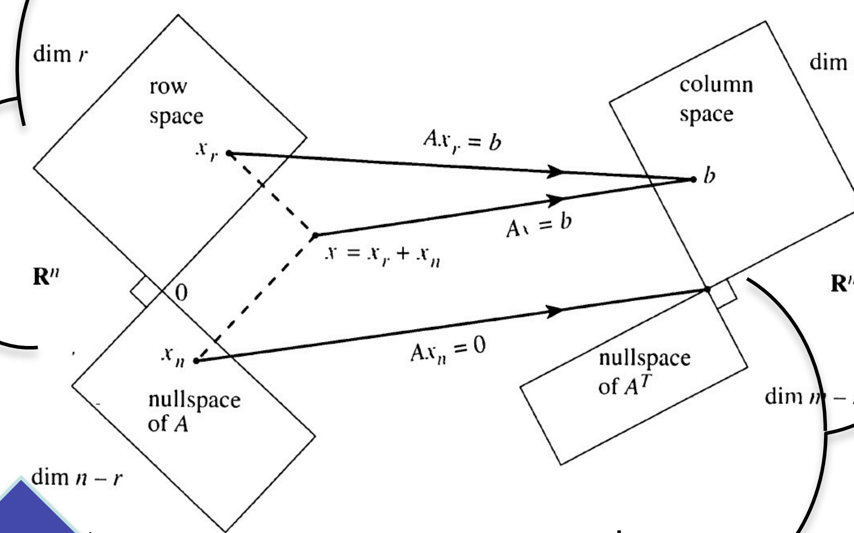
Linear Consensus

last time…

graph theory



matrix theory

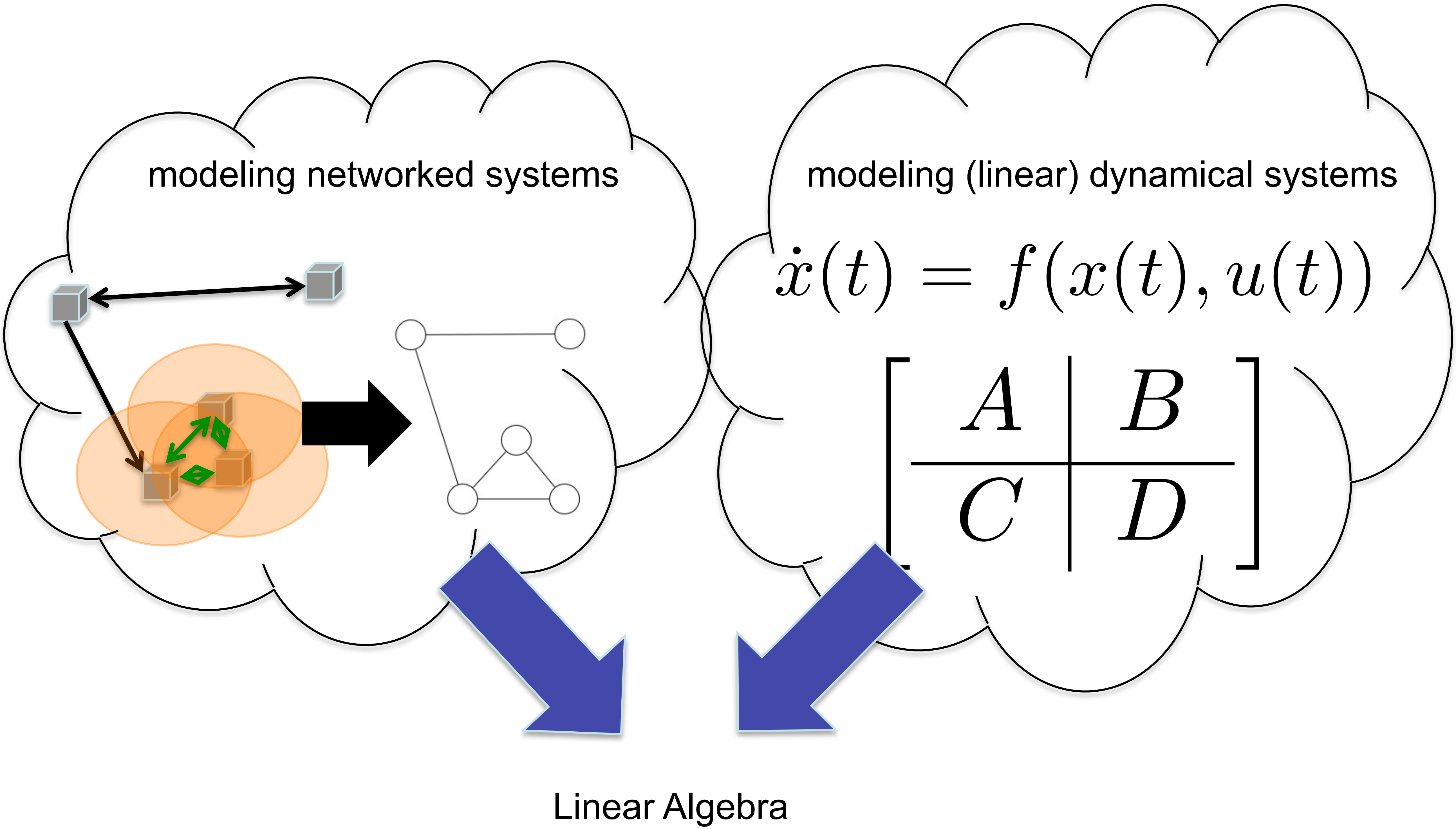


Algebraic Graph Theory

Theorem. *A graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$.*

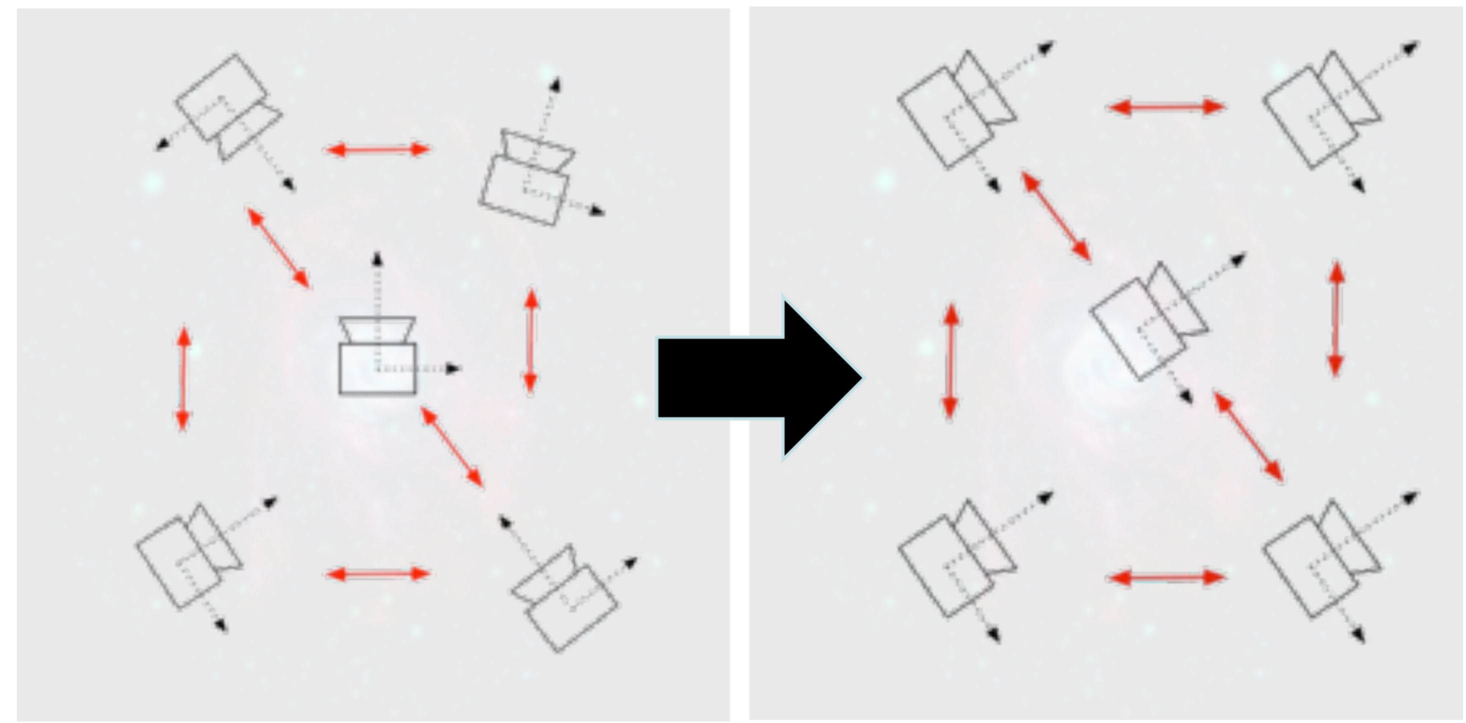
Theorem. *A matrix M is irreducible if and only if $\mathcal{G}(M)$ is strongly connected.*

Linear Algebra is the Key!



Linear Agreement

attitude consensus for satellites



sensing topology
induces a graph

$$\mathcal{G}(\mathcal{V},\mathcal{E})$$

satellites can sense
relative attitude

$$E(\mathcal{G})^T x(t)$$

distribute relative measurements
to generate control

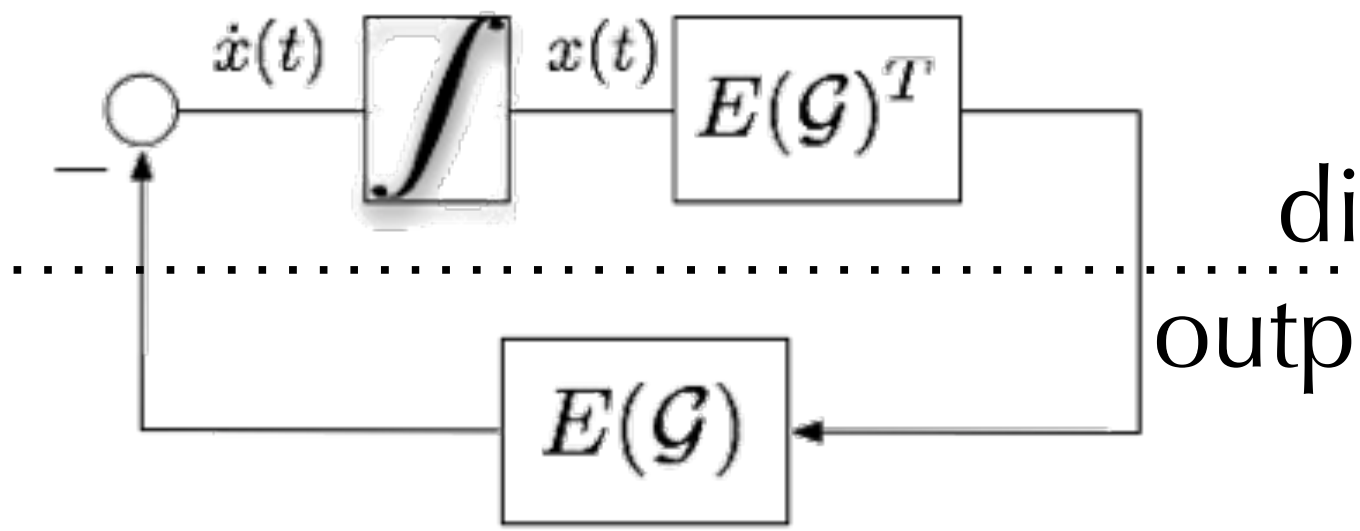
$$E(\mathcal{G})E(\mathcal{G})^T x(t)$$

Consensus Dynamics

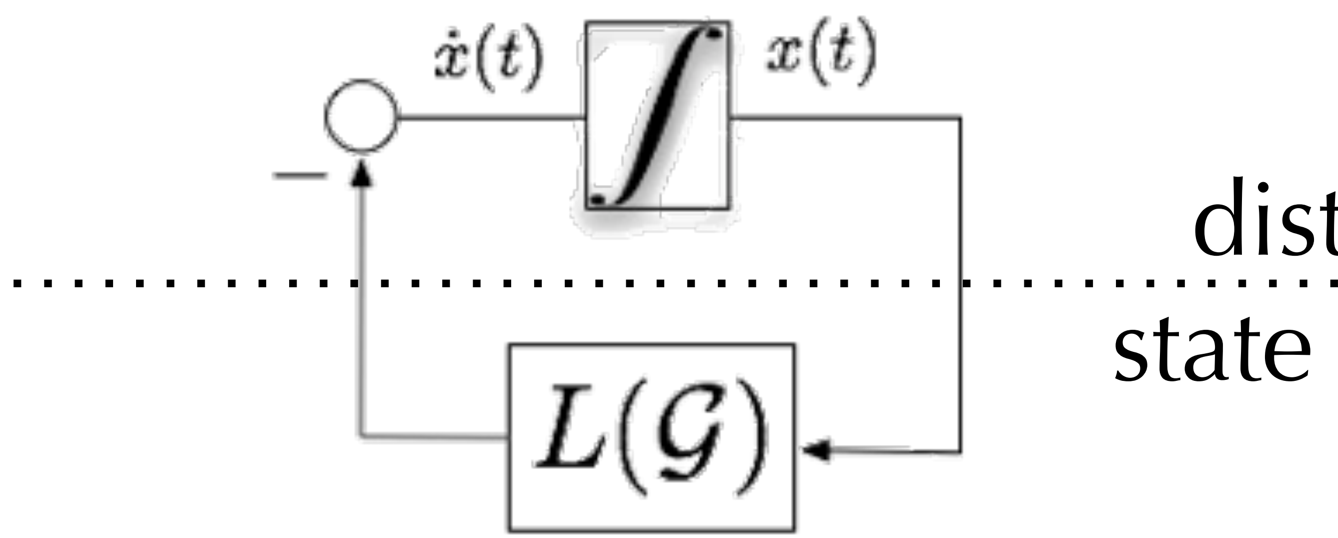
$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Linear Agreement

2 perspectives



distributed
output feedback



distributed
state feedback

Linear Agreement

Consensus Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

<u>analysis</u>

$$x(t) = e^{-L(\mathcal{G})t}x(0)$$

diagonalize $L(\mathcal{G}) = U\Lambda(\mathcal{G})U^T$

$$x(t) = \sum_{i=1}^{n-1} e^{-\lambda_{n-i+1}t}(u_i u_i^T)x(0) + \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0)$$

$$\lim_{t\to\infty} x(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0)$$

Linear Agreement

Consensus Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Definition

The *Agreement Set* $\mathcal{A} \subset \mathbb{R}^n$ is the subspace span$\{\mathbb{1}\}$,

$$\mathcal{A} = \{x \in \mathbb{R}^n \,|\, x_i = x_j, \forall\, i, j\}$$

$$\lim_{t \to \infty} x(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0) \in \mathcal{A}$$

Linear Agreement

Consensus Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Theorem

The linear agreement protocol converges to the agreement set from any initial condition if and only if $\lambda_2(\mathcal{G}) > 0$. Furthermore, $\lambda_2(\mathcal{G})$ dictates the rate of convergence.

Linear Agreement

Consensus Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

Corollary

The linear agreement protocol converges to the agreement set from any initial condition if and only if the underlying graph contains a spanning tree.

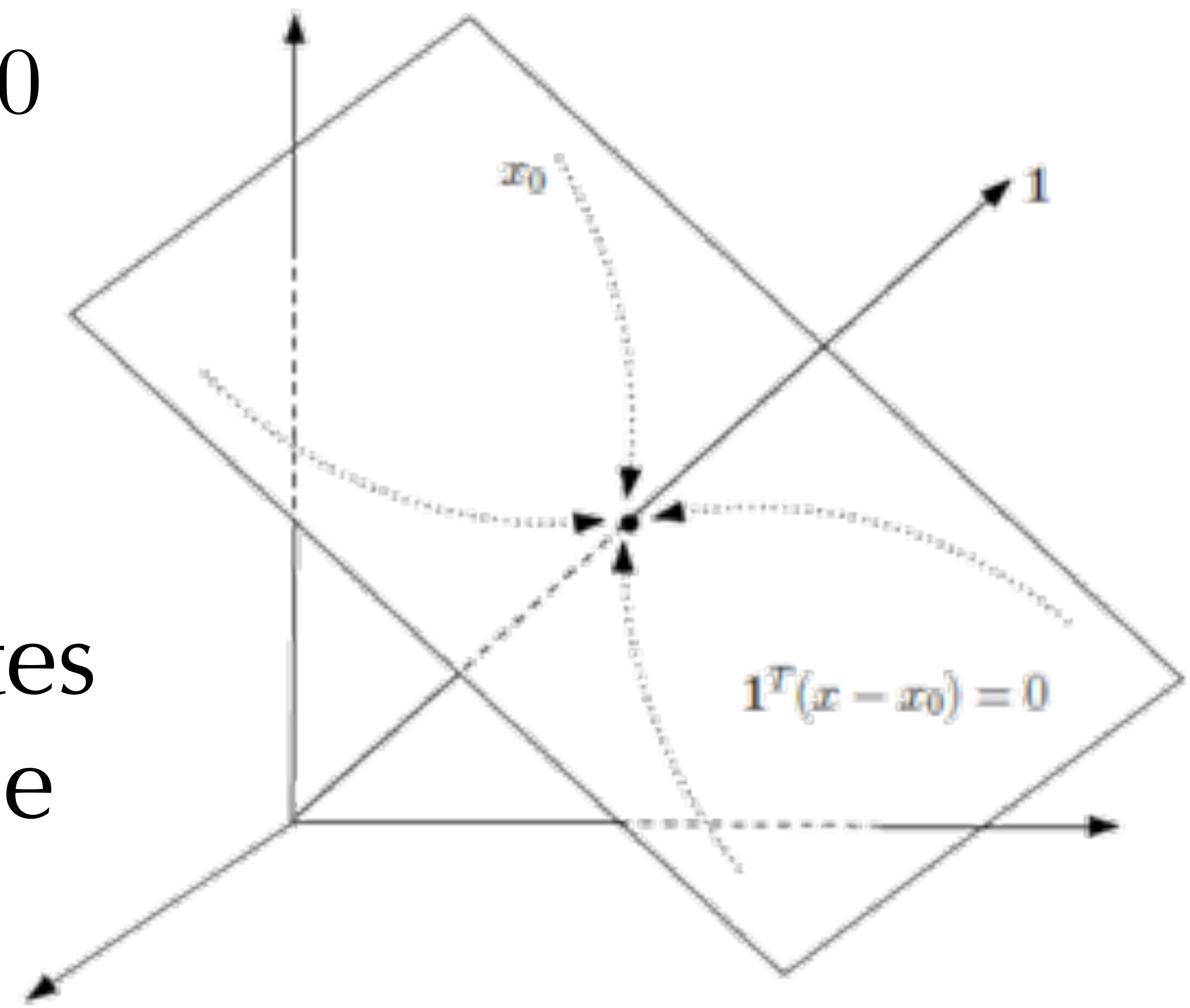
Linear Agreement

a *constant of motion* is a quantity that is conserved for all trajectories of a dynamical system

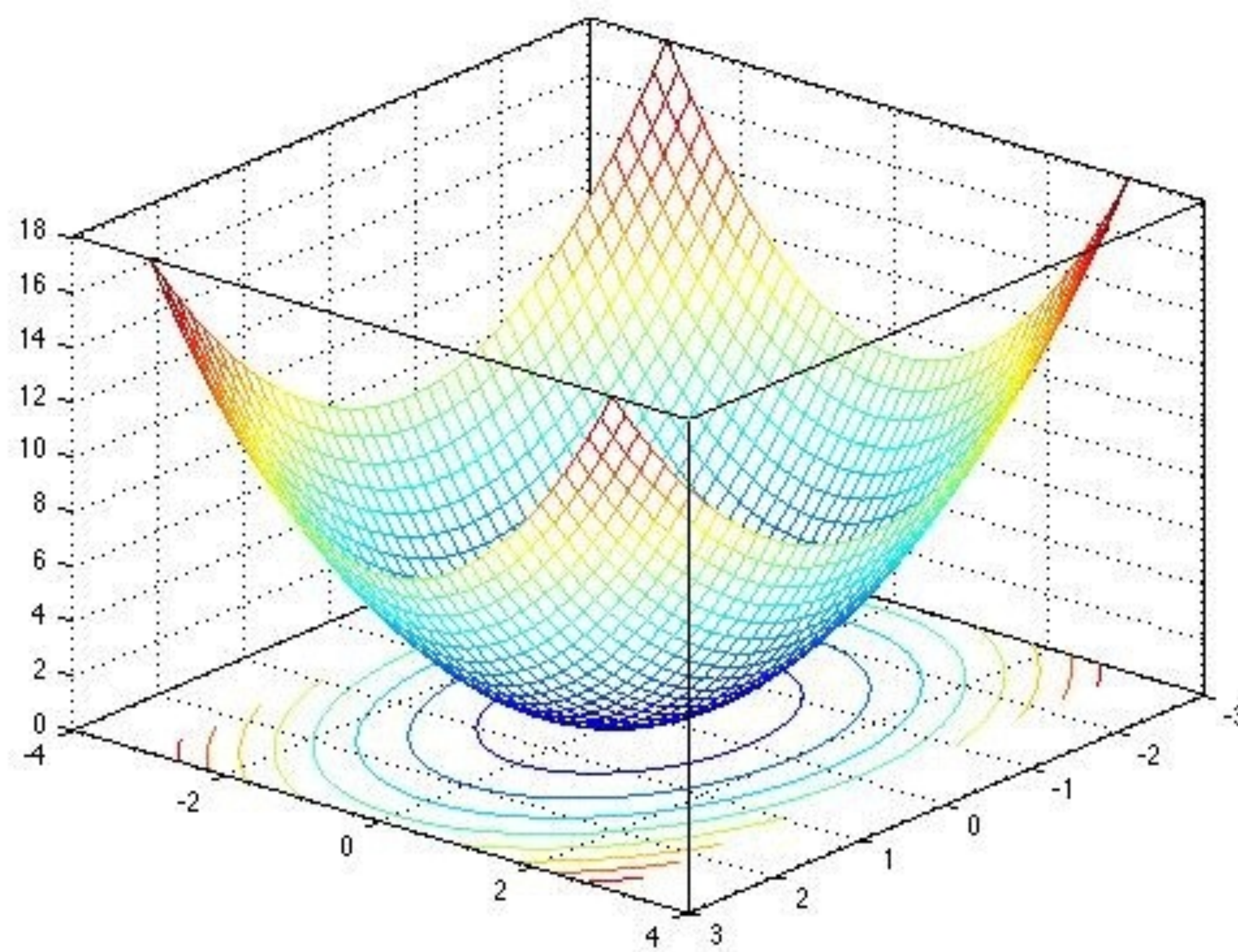
$$\frac{d}{dt}(\mathbf{1}^T x(t)) = -\mathbf{1}^T L(\mathcal{G}) x(t) = 0$$

$$\Rightarrow \quad \mathbf{1}^T(x(t) - x(0)) = 0, \ \forall t$$

the *centroid* of the system states is a constant of motion for the agreement protocol!



Gradient Dynamical Systems



Lyapunov Stability

(review)

consider an autonomous dynamical system

$$(1) \qquad \dot{x}(t) = f(x(t))$$

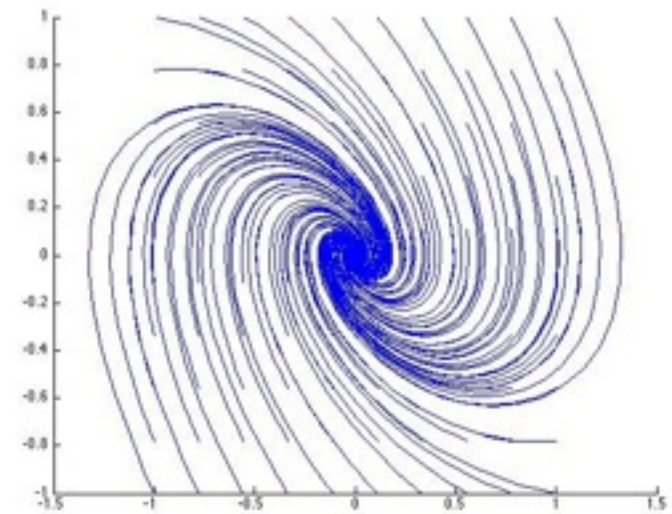
$f\ : W \to \mathbb{R}^n$

$W \subseteq \mathbb{R}^n$ is an open subset

$f \in \mathcal{C}^2$ twice differentiable

$x(t) \in \mathbb{R}^n$

phase portrait



a pendulum

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

Definition

The point $\overline{x} \in W$ is an *equilibrium point* of (1) if $f(\overline{x}) = 0$.

Lyapunov Stability

(review)

consider an autonomous dynamical system

$$(1) \quad \dot{x}(t) = f(x(t))$$

Theorem

Let $\overline{x} \in W$ be an equilibrium for (1). Let $V : U \to \mathbb{R}$ be a continuous function defined on a neighborhood $U \subset W$ of $\overline{x}$, differentiable on $U \setminus \{\overline{x}\}$, such that

(Lyapunov function)

(a) $V(\overline{x}) = 0$ and $V(x) > 0$ for all $x \in U$, $x \neq \overline{x}$,

(b) $\dot{V} \leq 0$ in $U \setminus \{\overline{x}\}$.

Then $\overline{x}$ is stable. Furthermore, if also

(*strict* Lyapunov function)

(c) $\dot{V} < 0$ in $U \setminus \{\overline{x}\}$,

then $\overline{x}$ is asymptotically stable.

Gradient Systems

$$(1) \quad \dot{x}(t) = f(x(t))$$

Consider a twice differentiable function $F : U \to \mathbb{R}^n$

$U \subset W \subseteq \mathbb{R}^n$

such that

$$f = -\nabla F = -\begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix}$$

Then

$$(2) \quad \dot{x}(t) = -\nabla F(x(t))$$

is called a *gradient dynamical system*

Gradient Systems

$$(2) \quad \dot{x}(t) = -\nabla F(x(t))$$

$U \subset W \subseteq \mathbb{R}^n$

Theorem

$\dot{F}(x) \leq 0$ for all $x \in U$ and $\dot{F}(x) = 0$ if and only if x is an equilibrium of (2)

Proof

chain rule

$$\begin{aligned} \frac{d}{dt}F(x) &= (\nabla F(x))^T \dot{x} \\ &= -(\nabla F(x))^T \nabla F(x) \\ &\leq 0 \end{aligned}$$

Gradient Systems

$$\dot{x}(t) = -\nabla F(x(t)) \tag{2}$$

Corollary

Let $\overline{x}$ be an isolated minimizer of F. Then $\overline{x}$ is an asymptotically stable equilibrium of (2).

Proof

isolated minimizer means $F(x) > F(\overline{x}), \ \forall x \neq \overline{x}$

verify that $F(x)$ is a strict Lyapunov function for (2)

Gradient Systems

what do gradient flows look like?

- look at “level surfaces” of $F(x)$

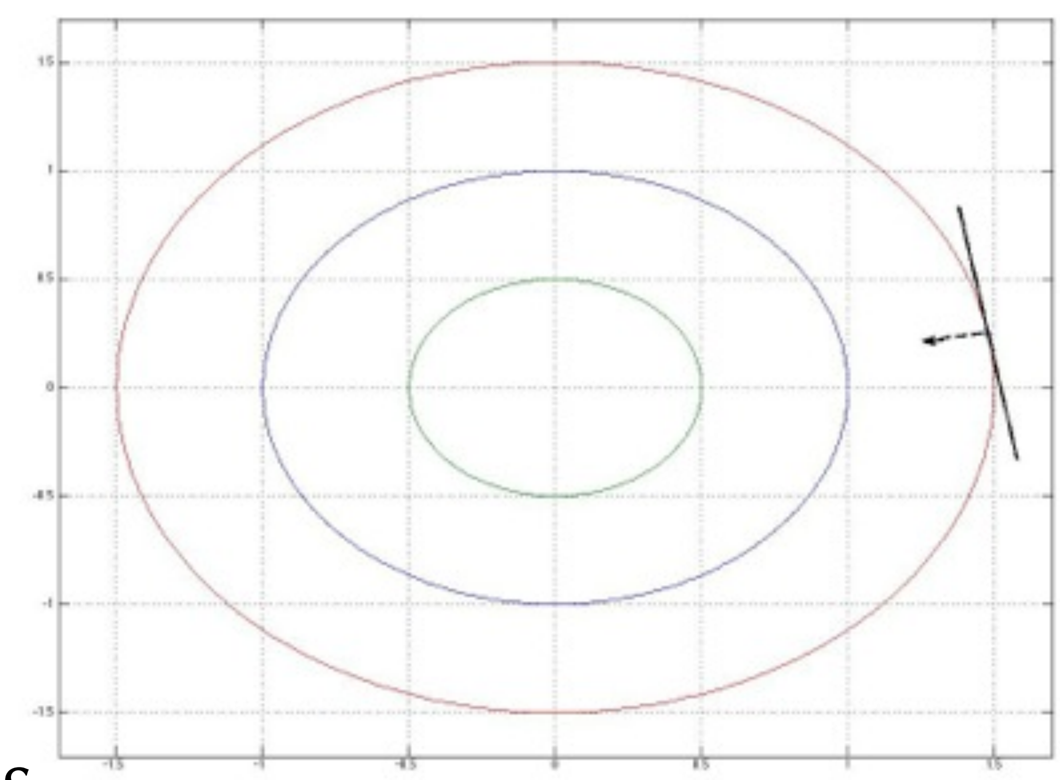
$$c \in \mathbb{R} \quad F^{-1}(c) = \{x \in \mathbb{R}^n \,|\, F(x) = c\}$$

example

$$F(x) = x_1^2 + x_2^2$$

$$\dot{x}(t) = -\nabla F(x(t))$$

$F(x) \neq 0$ a *regular point*



At *regular points* the vector field is perpendicular to the level surfaces

Gradient Systems

Theorem

Let

$$\dot{x}(t) = -\nabla F(x(t))$$

be a gradient system. At regular points the trajectories cross level surfaces orthogonally. Nonregular points are equilibria of the system. Isolated minima are asymptotically stable.

Gradient Systems

what if there are no *isolated* minimizers?

$$F(x) > F(\overline{x}), \forall\, x \neq \overline{x},\ \overline{x} \in \Omega$$

$$F(\overline{x}_1) = F(\overline{x}_2),\ \forall \overline{x}_1, \overline{x}_2 \in \Omega$$

does a gradient dynamical system still converge to a minimizer? which one?

Definition

ω-Limit Set

$$\Omega = \{a \in W \mid \exists t_n \to \infty \text{ with } x(t_n) \to a\}$$

Gradient Systems

$$\dot{x}(t) = -\nabla F(x(t))$$

Theorem

Let $z \in \Omega$ be an ω-limit point of a trajectory of a gradient flow. Then z is an equilibrium.
(stable)

Proof

$$x(t_n) \to z \Rightarrow F(x(t_n)) > F(z)$$

show invariance of Ω

$$\dot{F}(z) = 0, \ \forall \ z \in \Omega$$

Consensus as a Gradient System

$$F(x) = x^T L(\mathcal{G}) x$$

- symmetric matrix
- positive semi-definite
- convex function

$$\min_x F(x)$$

1st-Order Optimality Condition

$$\nabla F(x) = 0$$

function is minimized for any

$$x \in \mathcal{A}$$

Consensus as a Gradient System

$$F(x) = \frac{1}{2}x^T L(\mathcal{G})x$$

define gradient system

$$\begin{aligned}\dot{x}(t) &= -\nabla F(x(t)) \\ &= -L(\mathcal{G})x(t)\end{aligned}$$

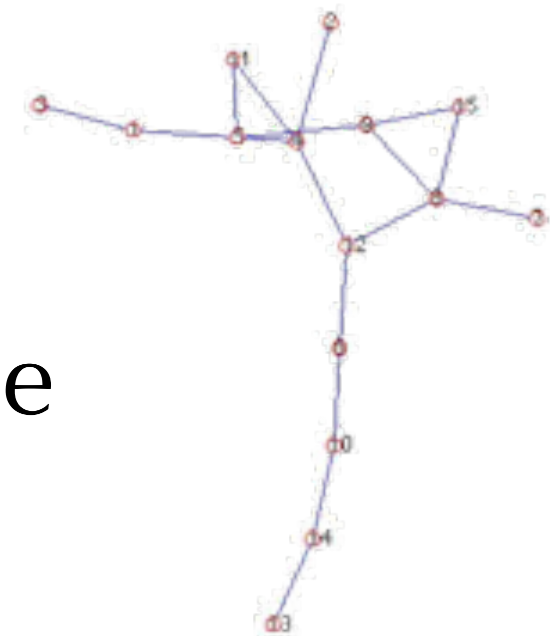
Theorem

Let $z \in \Omega$ be an ω-limit point of a trajectory of a gradient flow. Then z is an equilibrium.

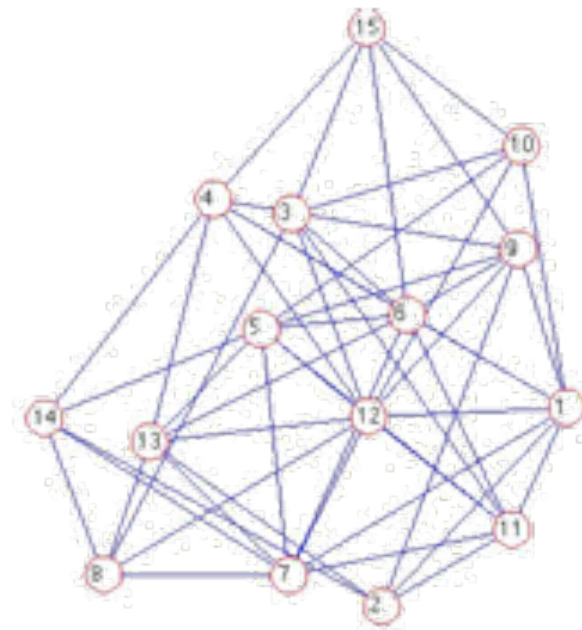
what is the ω-limit set?

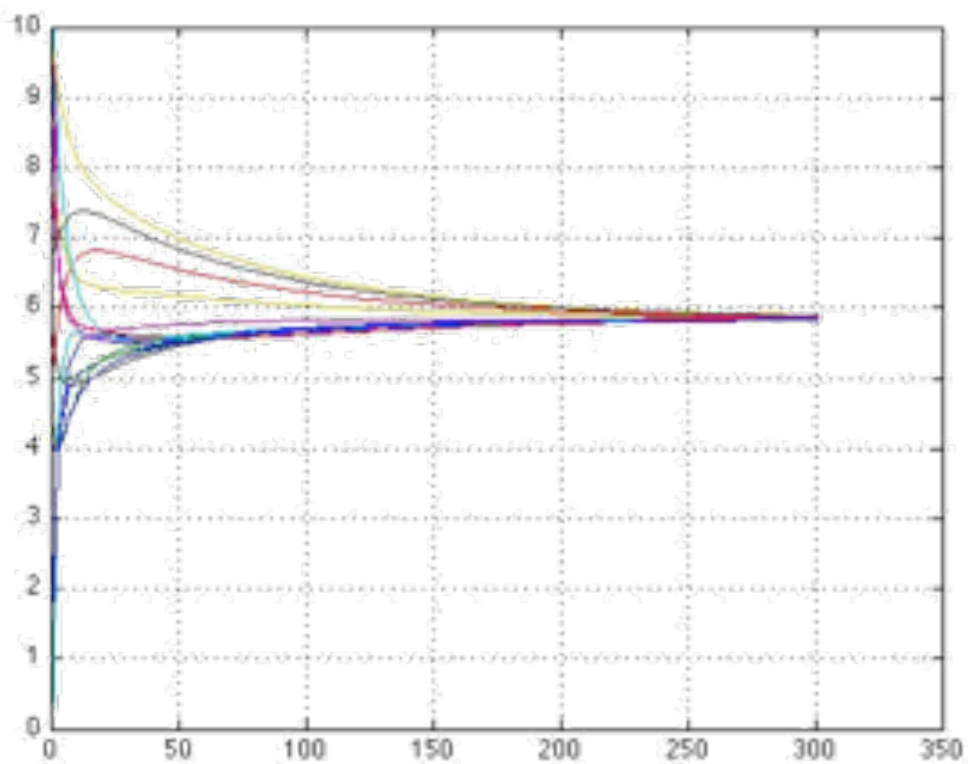
Linear Agreement

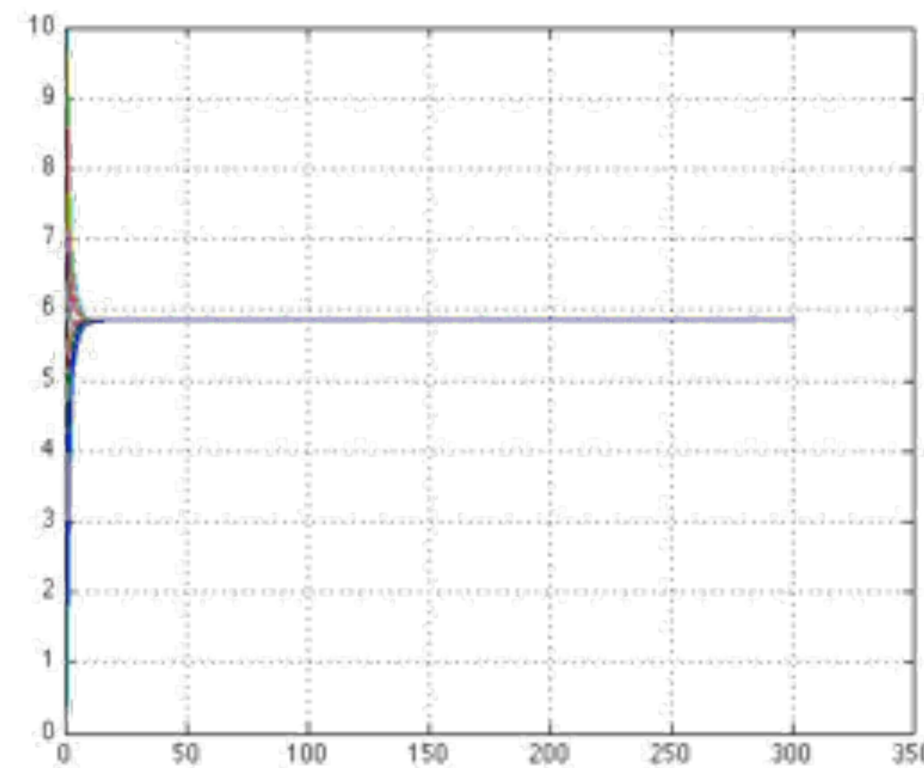
sparse



dense

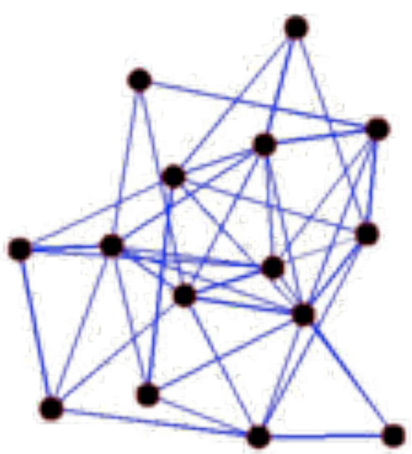


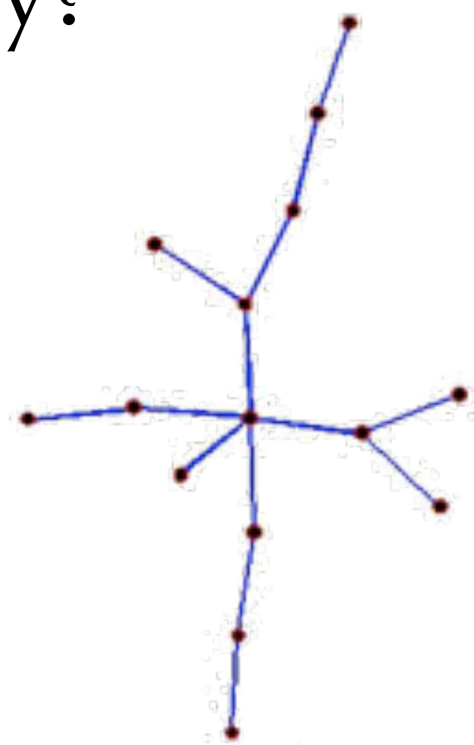


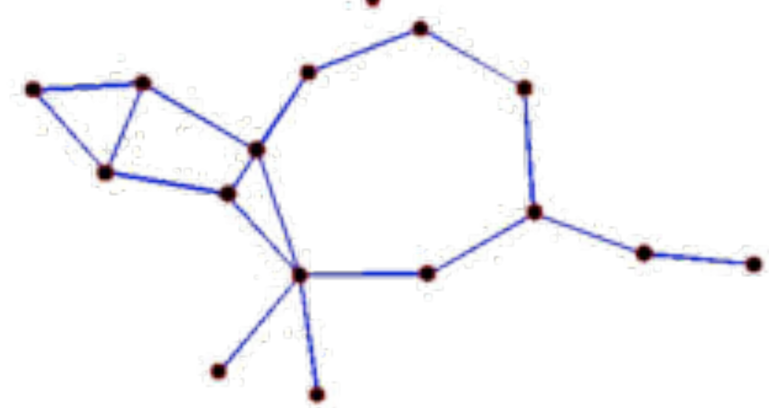


Linear Agreement

what can be said about trees, or graphs in general, and their algebraic connectivity?







which graph has largest algebraic connectivity?

why do we care?

Linear Agreement

Definition

A *vertex cut-set* for $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subset of $\mathcal{V}$ whose removal results in a disconnected graph. The *vertex connectivity* of $\mathcal{G}$, denoted $\kappa_v(\mathcal{G})$, is the cardinality of the smallest vertex cut-set of $\mathcal{G}$.

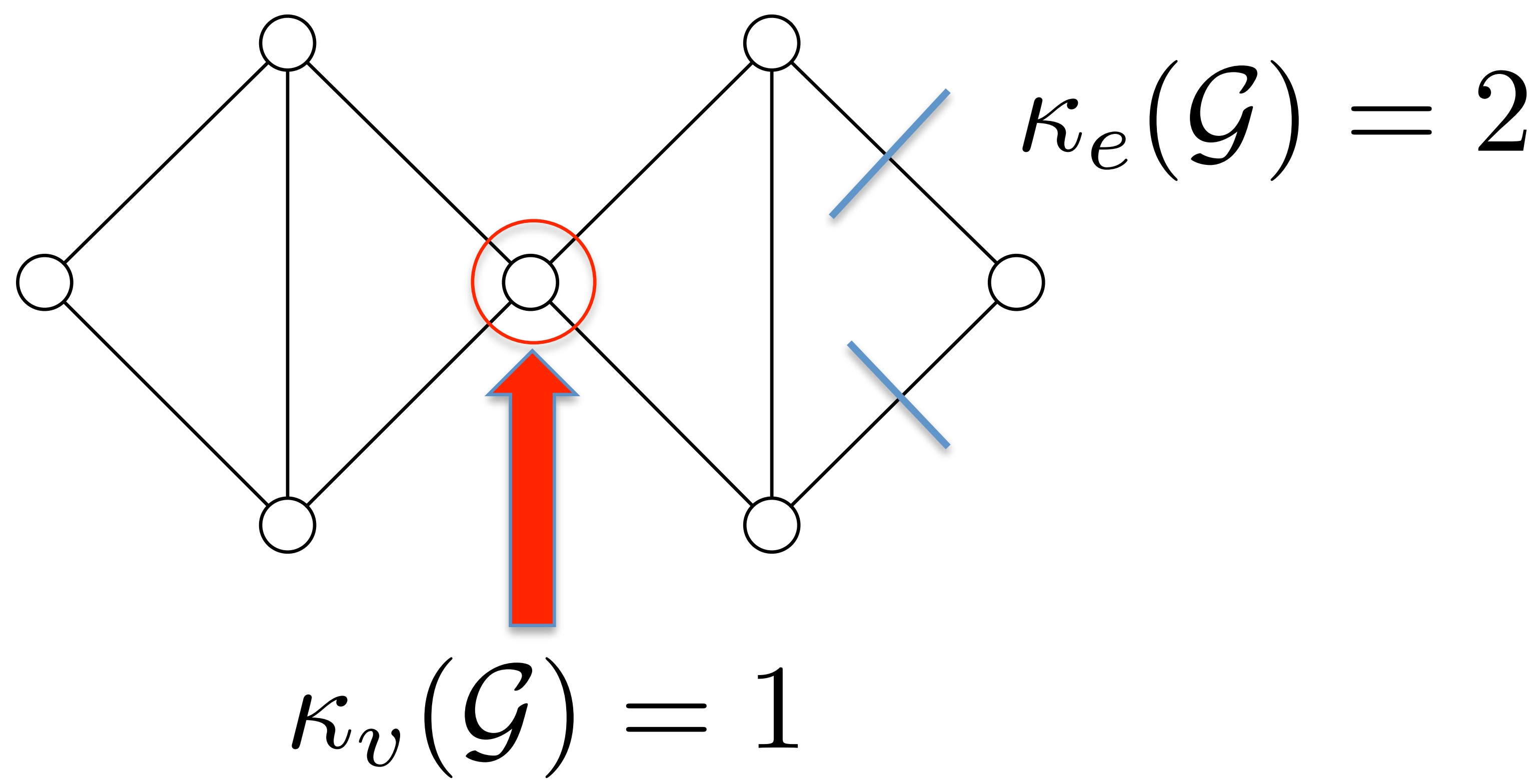
Definition

A *edge cut-set* for $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subset of $\mathcal{E}$ whose deletion increases the number of connected components of $\mathcal{G}$. The *edge connectivity* of $\mathcal{G}$, denoted $\kappa_e(\mathcal{G})$, is the cardinality of the smallest edge cut-set of $\mathcal{G}$.

Linear Agreement

some connectivity bounds…

$$\lambda_2(\mathcal{G}) \leq \kappa_v(\mathcal{G}) \leq \kappa_e(\mathcal{G}) \leq \min_i d_i$$



$$\lambda_2(\mathcal{G}) = 0.5858 \qquad \min_i d_i = 2$$