Analysis and Control of Multi-Agent Systems

Daniel Zelazo
Faculty of Aerospace Engineering
Technion-Israel Institute of Technology

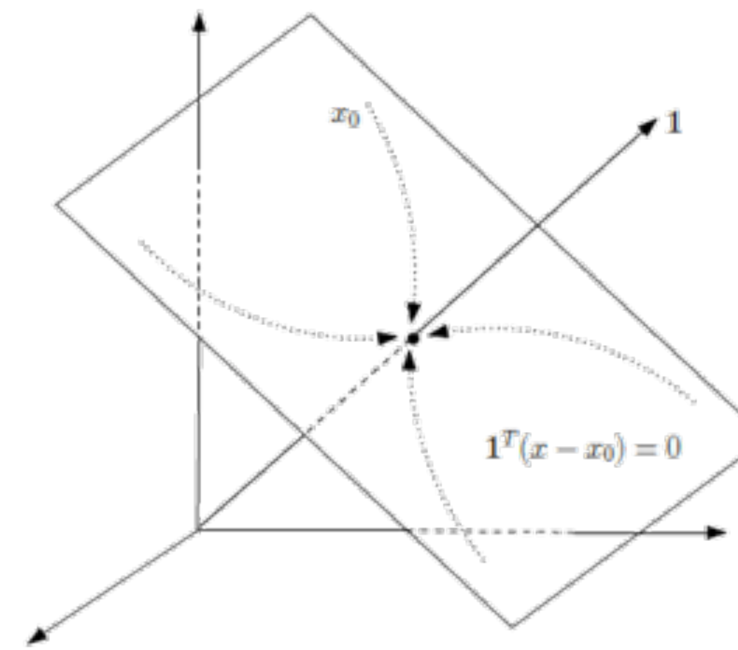
Linear Consensus

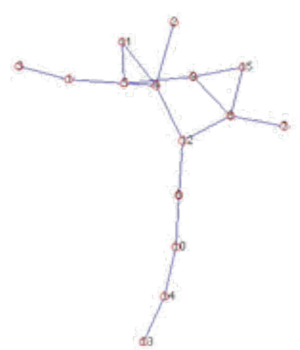
Weighted and Directed Graphs

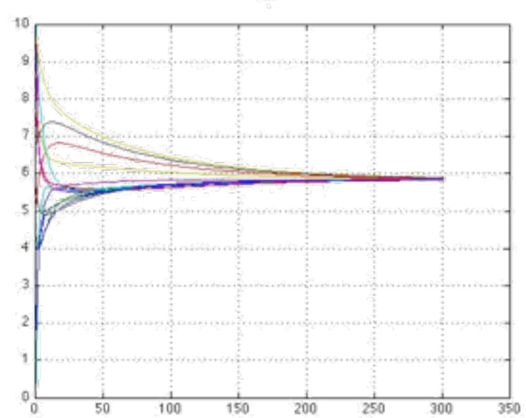
last time…

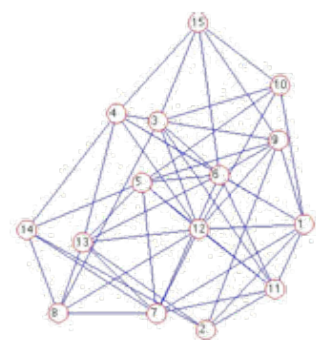
Linear Consensus

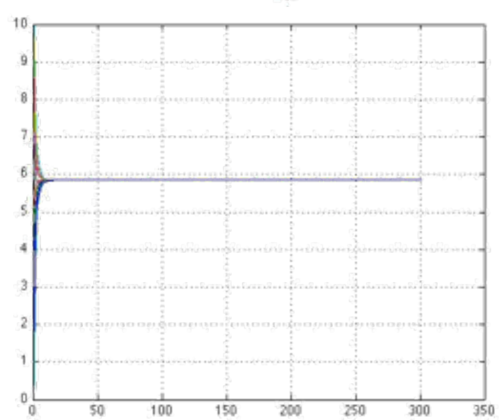
- undirected graphs
- continuous time
- gradient dynamics interpretation











$$\lim_{t \to \infty} x(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0) \in \mathcal{A}$$

$$\lambda_2(\mathcal{G})$$

Linear Agreement

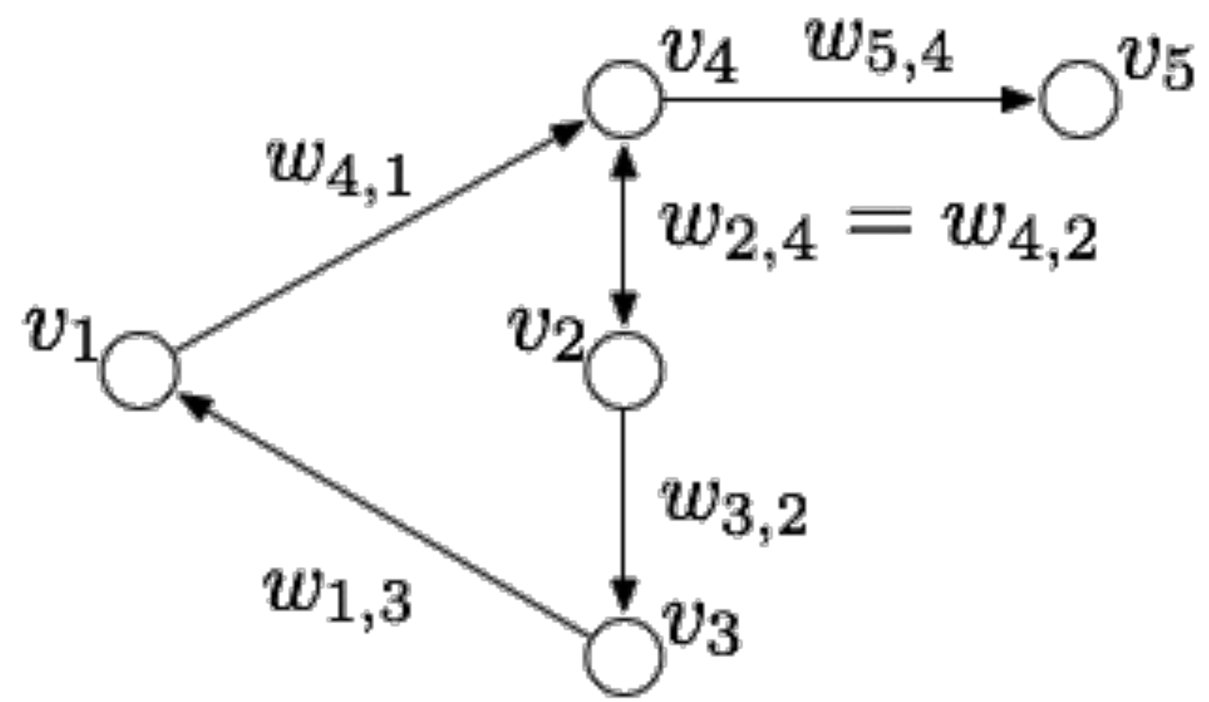
Linear Consensus over
weighted and directed graphs

$$\dot{x}_i(t) = \sum_{(i,j)\in\mathcal{E}} w_{ij}(x_j(t) - x_i(t))$$

- communication and sensing is uni-directional
- relative measurements are *weighted*

Linear Agreement

$$\dot{x}_i(t) = \sum_{(i,j)\in\mathcal{E}} w_{ij}(x_j(t) - x_i(t))$$



weighted and directed graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \to \mathbb{R}_{\geq 0}$$

$$w_{ij} = \mathcal{W}(e),\ e = (i,j) \in \mathcal{E}$$

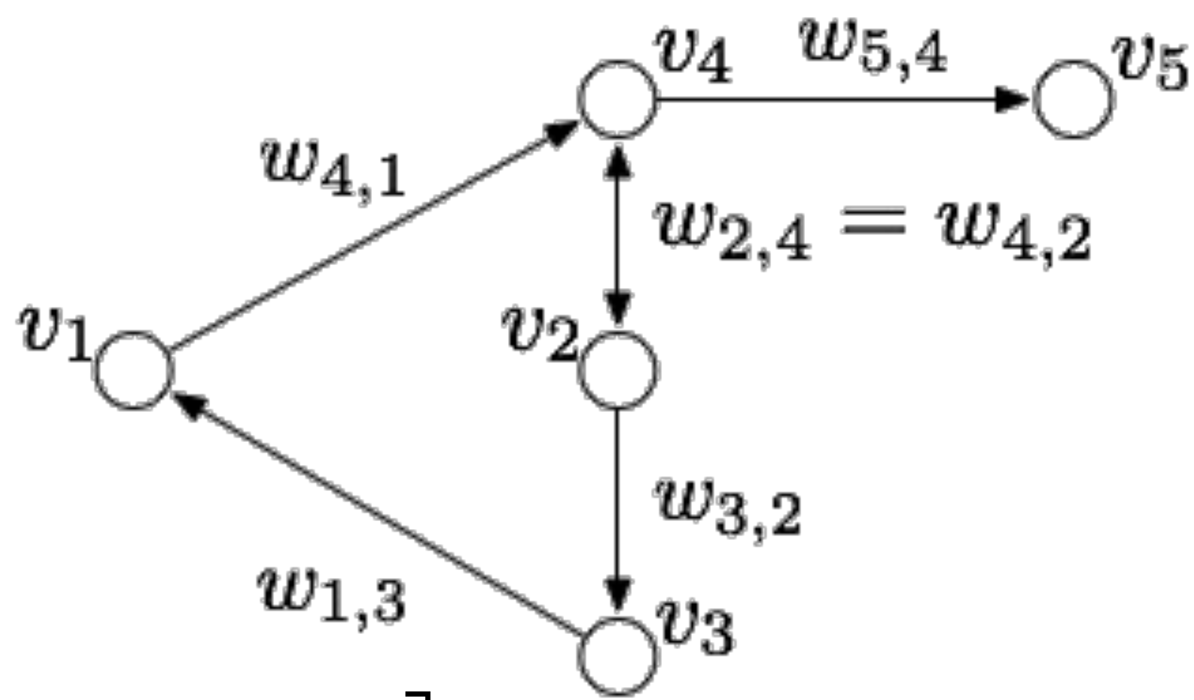
$$W = \mathbf{diag}[\mathcal{W}(e_1), \mathcal{W}(e_2), \cdots, \mathcal{W}(e_{|\mathcal{E}|})]$$ weight matrix

Linear Agreement

$$\dot{x}_i(t) = \sum_{(i,j)\in\mathcal{E}} w_{ij}(x_j(t) - x_i(t))$$

for example…

$$\begin{aligned}\dot{x}_4(t) \quad = \quad & w_{4,1}(x_1(t) - x_4(t)) \\ & +w_{4,2}(x_2(t) - x_4(t))\end{aligned}$$



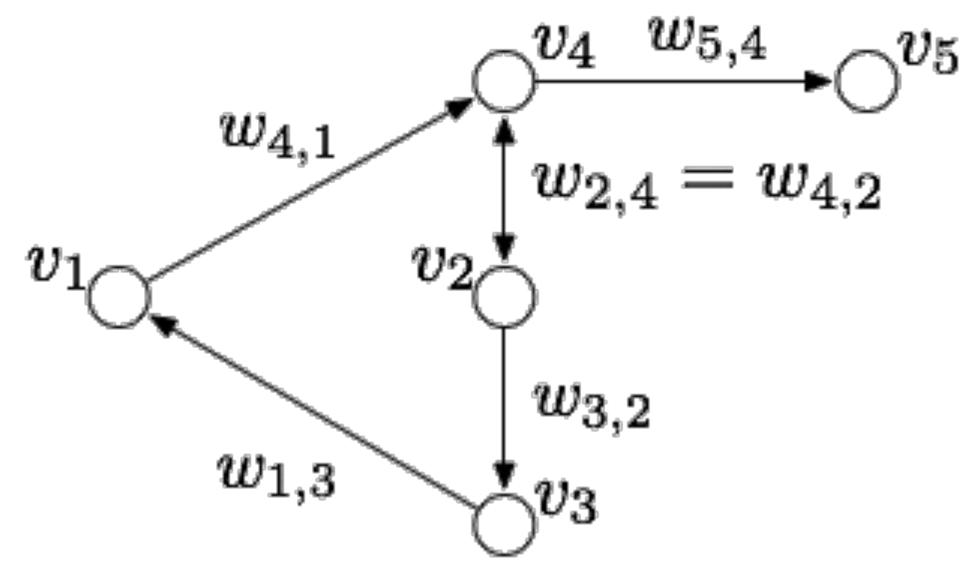
a *directed* and *weighted* Laplacian dynamics

$$\dot{x}(t) = -\begin{bmatrix} w_{1,3} & 0 & -w_{1,3} & 0 & 0 \\ 0 & w_{2,4} & 0 & -w_{2,4} & 0 \\ 0 & -w_{3,2} & w_{3,2} & 0 & 0 \\ -w_{4,1} & -w_{4,2} & 0 & w_{4,1} + w_{4,2} & 0 \\ 0 & 0 & 0 & -w_{5,4} & w_{5,4} \end{bmatrix} x(t)$$

Linear Agreement

The "in"-degree graph Laplacian Matrix

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$



weighted adjacency matrix

$$[A(\mathcal{G})]_{ij} = \begin{cases} w_{i,j}, & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

in-degree matrix

$$[\Delta_{in}(\mathcal{G})]_{ii} = \sum_{\{j | (v_j, v_i) \in \mathcal{E}\}} w_{ij}$$

$$L_{in}(\mathcal{G}) = \Delta_{in}(\mathcal{G}) - A(\mathcal{G})$$

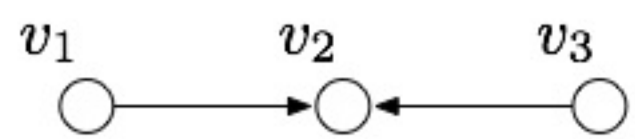
Linear Agreement

The "in"-degree graph Laplacian Matrix

$$L_{in}(\mathcal{G}) = \Delta_{in}(\mathcal{G}) - A(\mathcal{G})$$

some properties…

$$\mathbf{1} \in \mathcal{N}(L(\mathcal{G}))$$



$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{N}(L(\mathcal{G})) = \textbf{span}\left\{\mathbf{1}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

we need to refine our notions of trees, cycles, and connectedness to better understand the directed Laplacian

Linear Agreement

The "in"-degree graph Laplacian Matrix

$$L_{in}(\mathcal{G}) = \Delta_{in}(\mathcal{G}) - A(\mathcal{G})$$

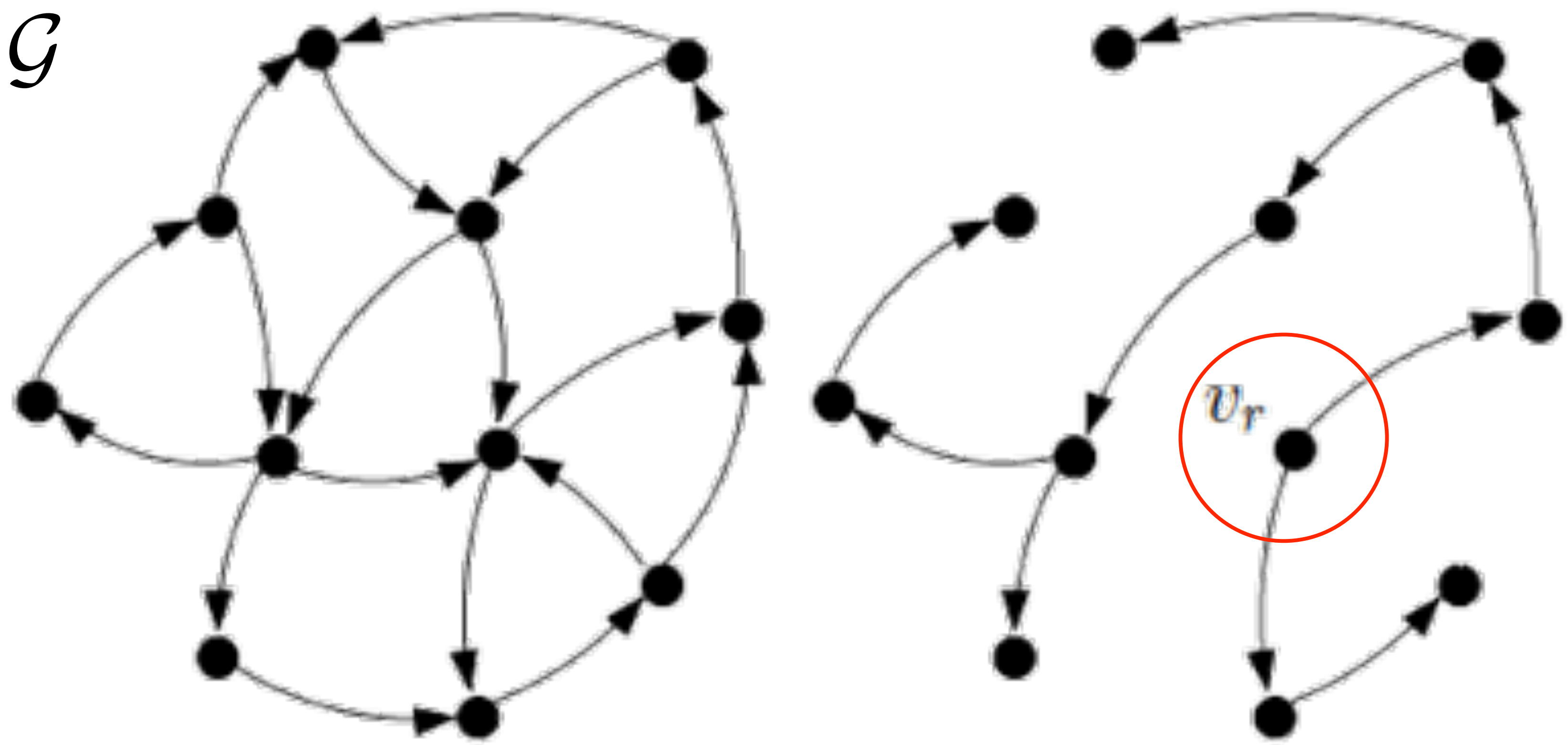
Definition

A *rooted out-tree* (*arborescence*) is an acyclic (no directed cycle) with a node $r \in \mathcal{V}$ (the root) such that

1. there is a directed path from r to every other node in $\mathcal{V}$,
2. the in-degree of r is zero, and
3. the in-degree of every other vertex is one.

Linear Agreement

$\mathcal{G}$



(a) Original Graph (b) Rooted Out-Branching Subgraph

$\mathcal{G}$ contains a rooted out-branching

Linear Agreement

Theorem

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ contains a rooted out-branching if and only if $\mathbf{rank}\, L(\mathcal{G}) = n - 1$.

proof

by construction $L(\mathcal{G})\mathbf{1} = 0$

$\mathbf{rank}\, L(\mathcal{G}) = n - 1 \Leftrightarrow$ 0 is a simple eigenvalue

look at characteristic polynomial

$$p_{\mathcal{G}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 \quad (\alpha_0 \to 0)$$

$$\alpha_1 = \sum_v \det L_v(\mathcal{G})$$

Matrix-Tree Theorem for Weighted Graphs

Linear Agreement

Theorem

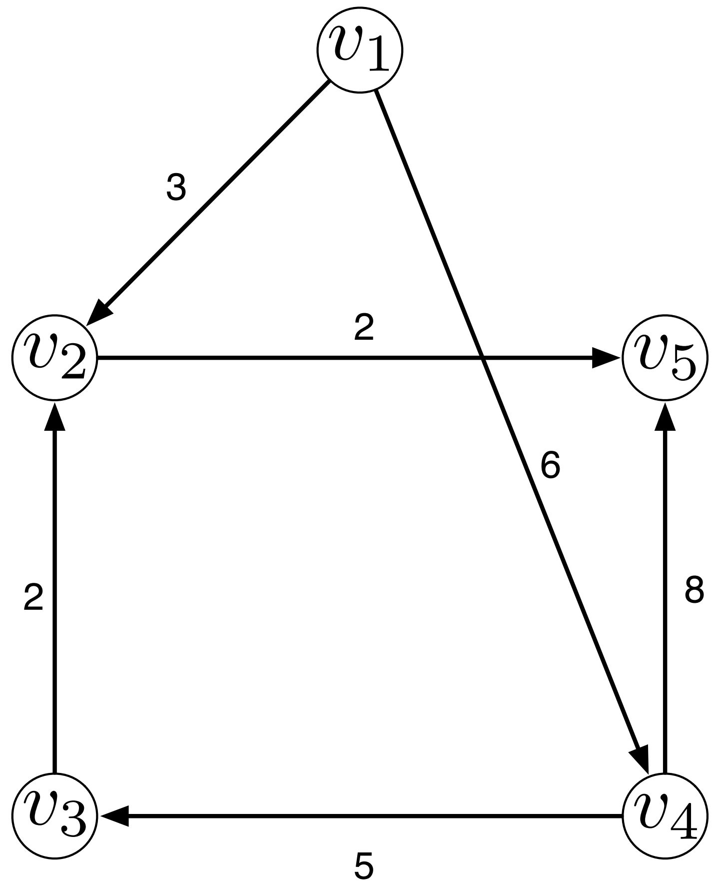
Let $v \in \mathcal{V}$ be an arbitrary vertex of a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. Then

$$\mathbf{det}\, L_v(\mathcal{G}) = \sum_{T \in \mathcal{T}_v} \prod_{e \in T} \mathcal{W}(e),$$

where $\mathcal{T}_v$ is the set of rooted out-branchings with root v in $\mathcal{G}$, $\prod_{e \in T} \mathcal{W}(e)$ is the product of the weights on the edges of an out-branching T, and $L_v(\mathcal{G})$ is the v-th principal sub-matrix of $L(\mathcal{G})$.

Linear Agreement

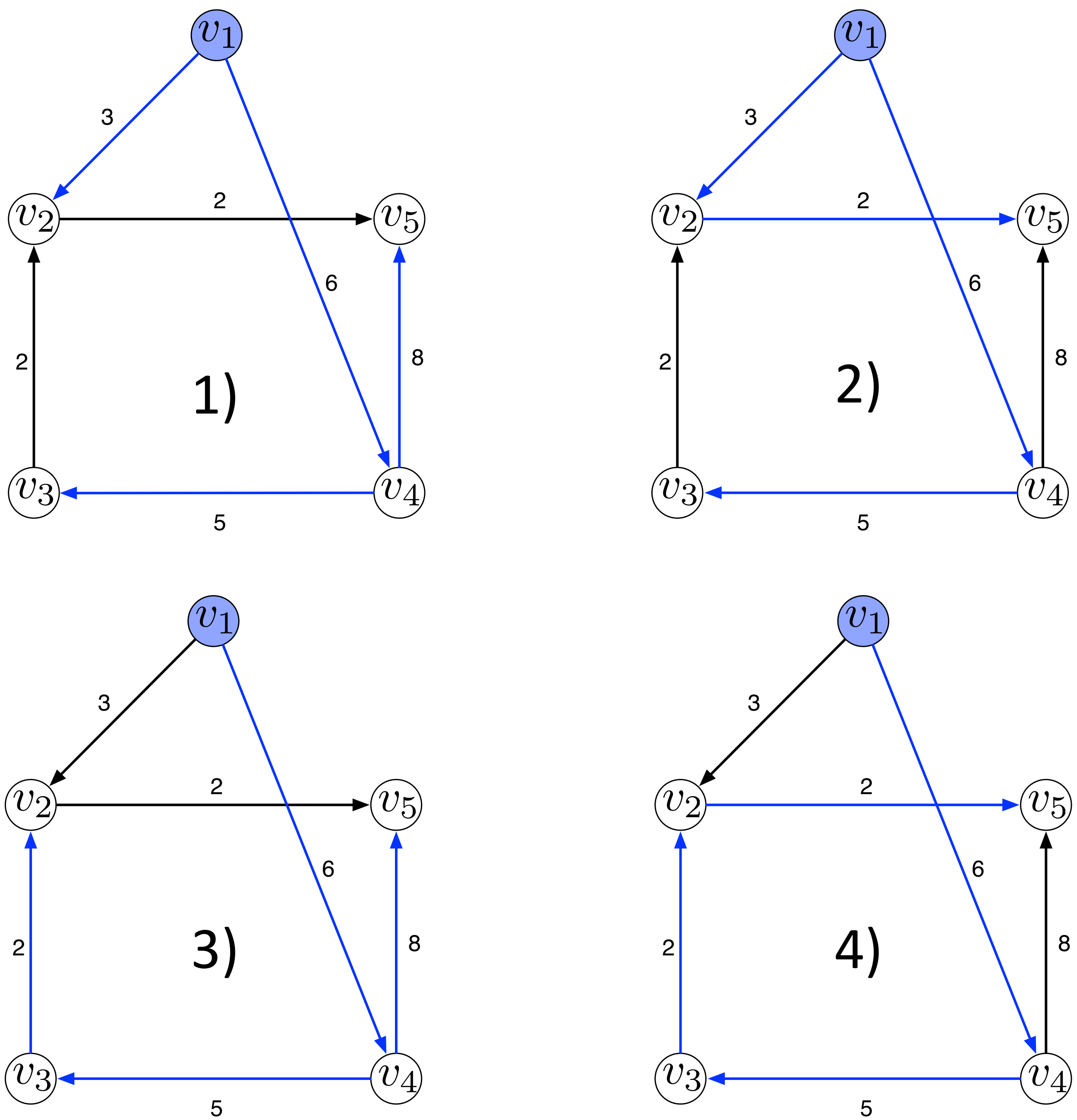
Matrix-Tree Theorem for Weighted Graphs
example…



$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -3 & 5 & -2 & 0 & 0 \\ 0 & 0 & 5 & -5 & 0 \\ -6 & 0 & 0 & 6 & 0 \\ 0 & -2 & 0 & -8 & 10 \end{bmatrix}$$

Linear Agreement

Matrix-Tree Theorem for Weighted Graphs
example...



1) $\Pi_{e \in \mathcal{T}_{v_1}} w(e) = 720$

2) $\Pi_{e \in \mathcal{T}_{v_1}} w(e) = 180$

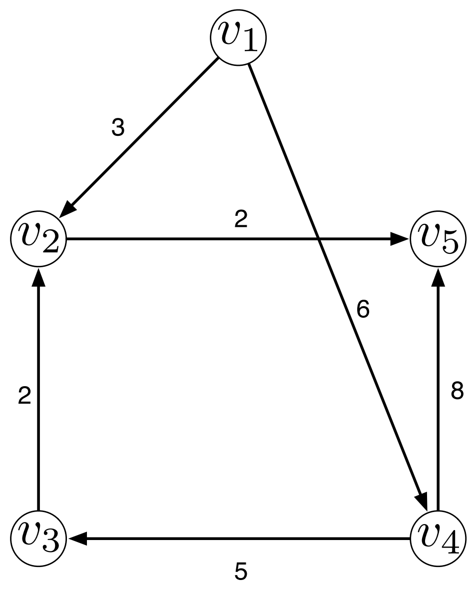
3) $\Pi_{e \in \mathcal{T}_{v_1}} w(e) = 480$

4) $\Pi_{e \in \mathcal{T}_{v_1}} w(e) = 120$

$\Sigma_{T \in \mathcal{T}_{v_1}} \Pi_{e \in T} w(e) = 1500$

Linear Agreement

Matrix-Tree Theorem for Weighted Graphs
example…



$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -3 & 5 & -2 & 0 & 0 \\ 0 & 0 & 5 & -5 & 0 \\ -6 & 0 & 0 & 6 & 0 \\ 0 & -2 & 0 & -8 & 10 \end{bmatrix}$$

principal minor at node v_1

$$L_{v_1}(\mathcal{G}) = \begin{bmatrix} 5 & -2 & 0 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 6 & 0 \\ -2 & 0 & -8 & 10 \end{bmatrix} \qquad \det(L_{v_1}(\mathcal{G})) = 1500$$

$$\det(\lambda I - L(\mathcal{G})) = \lambda^5 - 26\lambda^4 + 245\lambda^3 - 1000\lambda^2 + 1500\lambda$$

Linear Agreement

The "in"-degree graph Laplacian Matrix

$$L_{in}(\mathcal{G}) = \Delta_{in}(\mathcal{G}) - A(\mathcal{G})$$

where are the eigenvalues?

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

$$x(t) = e^{-L(\mathcal{G})t}x(t_0)$$

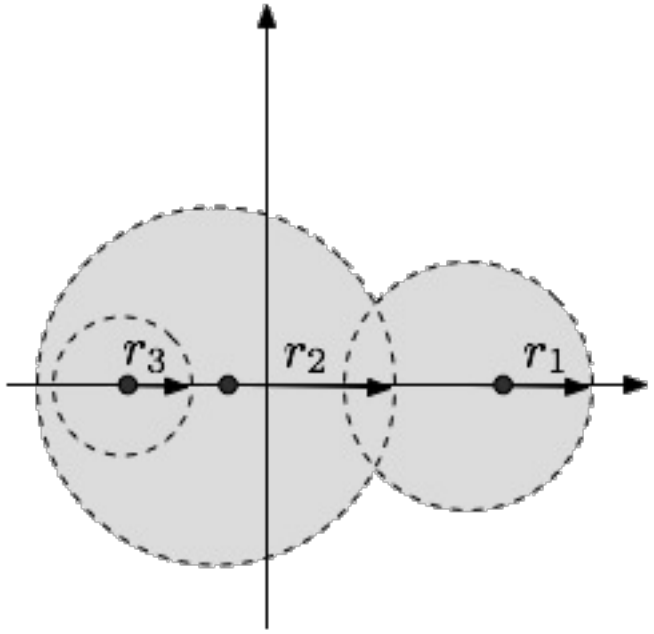
$$\lim_{t\to\infty} x(t) = ???$$

Linear Agreement

Theorem [Geršgorin Circle Theorem]

Consider a square matrix $M \in \mathbb{R}^{n \times n}$. Let $D([M]_{ii}, r_i)$ be a closed disc on the complex plane, centered at $[M]_{ii}$ with radius $r_i = \sum_{i \neq j} |[M]_{ij}|$. Then the eigenvalues of M lie in the union of the discs,

$$\lambda(M) \subseteq \cup_i D([M]_{ii}, r_i).$$



Linear Agreement

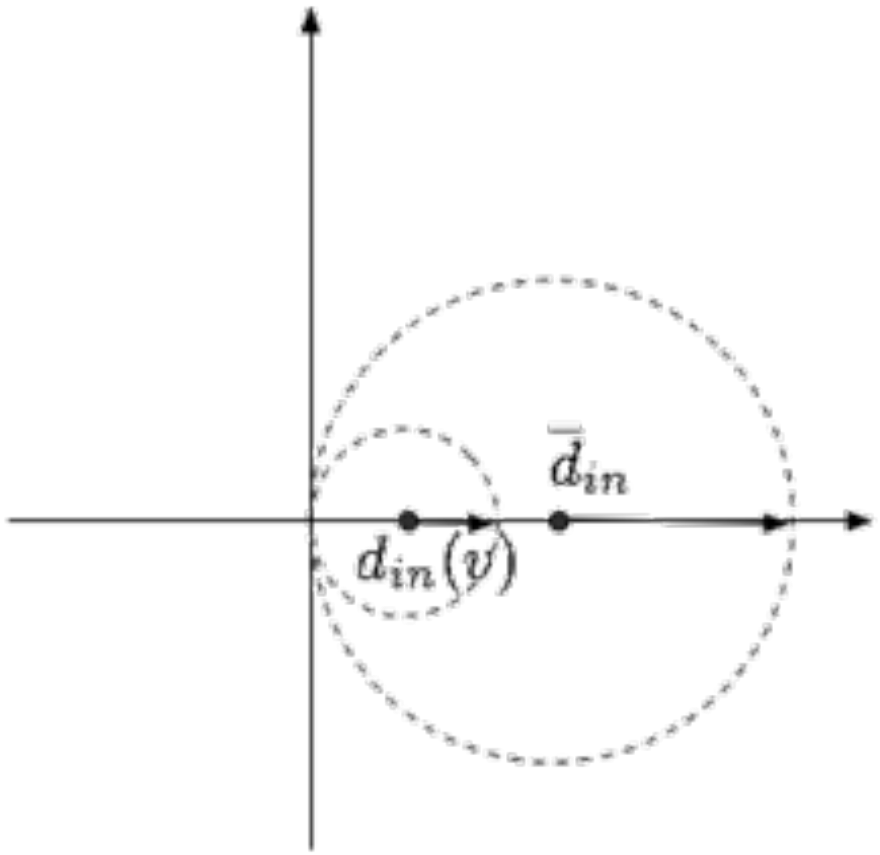
Theorem

The eigenvalues of the graph Laplacian for a weighted and directed graph $\mathcal{G}$ have non-negative real parts.

proof

Geršgorin Circle Theorem

all Geršgorin discs are contained in the closed right-half plane



Linear Agreement

Consensus Dynamics

$$\dot{x}(t) = -L(\mathcal{G})x(t)$$

recall Jordan form: $$L(\mathcal{G}) = PJ(\Lambda)P^{-1}$$

$$J(\Lambda) = \begin{bmatrix} J(0) & & & \\ & J(\lambda_2) & & \\ & & \ddots & \\ & & & J(\lambda_n) \end{bmatrix}$$

each block is the size of the algebraic multiplicity of corresponding eigenvalue

note: there may not be n Jordan blocks

$$J(\lambda_i) = \lambda_i I + \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

← nilpotent matrix

Linear Agreement

Theorem

For a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ containing a rooted out-branching, the state trajectory generated by $\dot{x}(t) = -L(\mathcal{G})x(t)$ with initial condition $x(0)$ satisfies

$$\lim_{t\to\infty} x(t) = (p_1 q_1^T)x(0),$$

where p_1 and q_1 are, respectively, the right- and left-eigenvectors associated with the zero eigenvalue of $L(\mathcal{G})$, normalized such that $p_1^T q_1 = 1$.

in particular... $\lim_{t\to\infty} x(t) = (q_1^T x_0)\mathbf{1}$

Linear Agreement

Corollary

The consensus protocol over a weighted and directed graph converges to the agreement set for all almost all initial conditions if and only if the digraph contains a rooted out-branching.

What about *average agreement?*

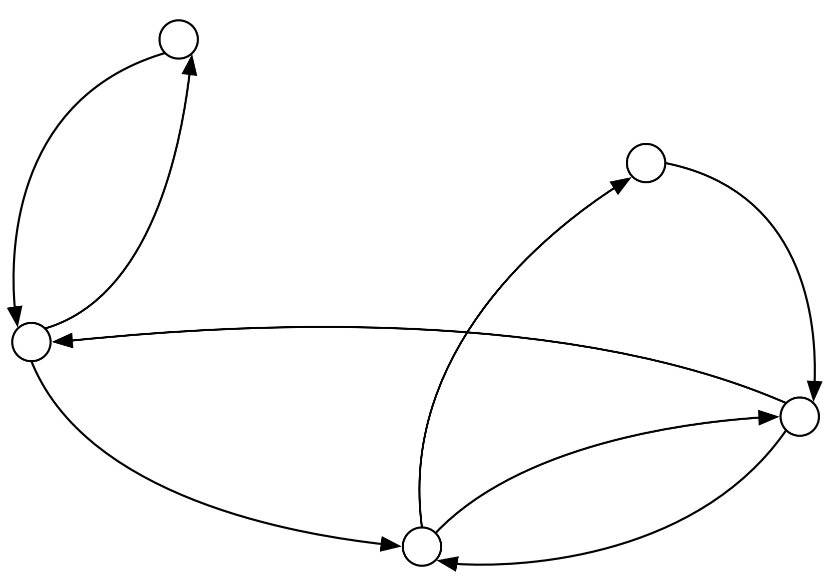
Can the consensus protocol over directed graphs reach average agreement?

If yes, what are the right conditions?

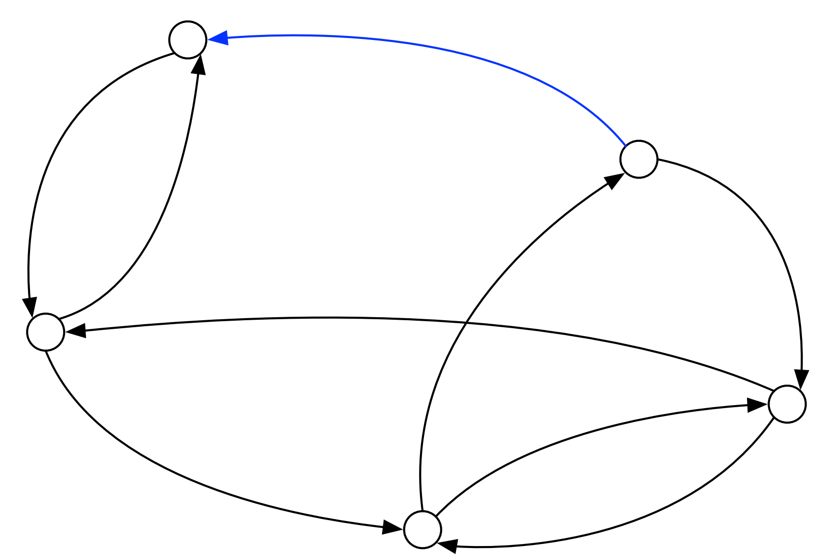
Linear Agreement

Definition

A digraph is *balanced* if for every vertex, the in-degree and the out-degree are equal.



balanced



unbalanced

Linear Agreement

Proposition

A balanced digraph is weakly connected if and only if it is strongly connected.

proof

balanced and strongly connected $\Rightarrow$ weakly connected (trivial)

balanced and weakly connected $\Rightarrow$ strongly connected

partition node set such that induced subgraph is strongly connected

$\mathcal{V} = S_1 \cup S_2 \cup \cdots \cup S_r$ (single vertex ok)

$\mathcal{G}_{S_i} \subset \mathcal{G}$ strongly connected

Linear Agreement

Proposition

A balanced digraph is weakly connected if and only if it is strongly connected.

proof

graph is balanced

$$\sum_{v \in S_k} d_{in}(v) = \sum_{v \in S_k} d_{out}(v) \Rightarrow \sum_{v \in S_k} d_{in}(v) - \sum_{v \in S_k} d_{out}(v) = 0$$

\# edges entering S_k $-$ # edges leaving S_k

number of edges entering each component equals number of edges leaving

Linear Agreement

Proposition

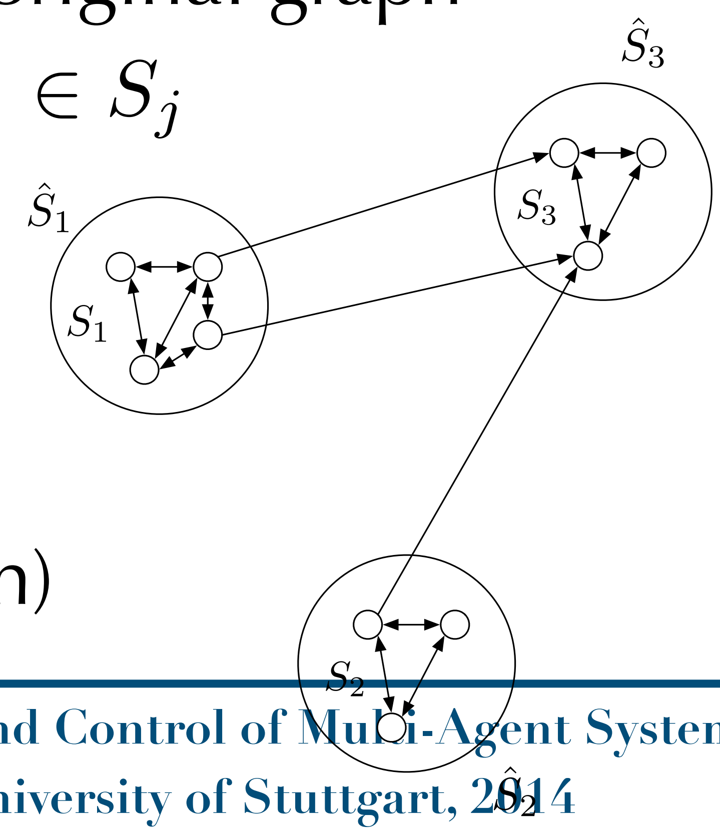
A balanced digraph is weakly connected if and only if it is strongly connected.

proof

define a new graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$

$\mathcal{V}' = \{\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_r\}$ each vertex corresponds to connected component defined in original graph

$$e' = (\hat{S}_i, \hat{S}_j) \in \mathcal{E}' \Leftrightarrow \exists e = (v_i, v_j) \in \mathcal{E} \; s.t. \; v_i \in S_i, \; v_j \in S_j$$



$\underline{r > 1}$

$$d_{in}(\hat{S}_i) = 0 \Rightarrow d_{out}(\hat{S}_i) = 0$$

$\Rightarrow$ graph is not weakly connected! (contradiction)

Linear Agreement

Proposition

A balanced digraph is weakly connected if and only if it is strongly connected.

proof

$$d_{in}(\hat{S}_i) > 0 \Rightarrow d_{out}(\hat{S}_i) > 0,\ k = 1, \ldots, r$$

$\Rightarrow$ graph must contain a cycle (why?)! we can partition into smaller number of connected components

$\Rightarrow$ $r = 1$ there can only be one component, i.e., the graph is strongly connected

Linear Agreement

Proposition

A graph Laplacian of balanced digraph containing a rooted out-branching satisfies

$$L_{in}(\mathcal{G})\mathbf{1} = 0 \text{ and } \mathbf{1}^T L_{in}(\mathcal{G}) = 0^T$$

$$L_{in}(\mathcal{G}) = \Delta_{in}(\mathcal{G}) - A(\mathcal{G})$$

proof

$$L_{in}(\mathcal{G}) + L_{in}(\mathcal{G})^T = L(\hat{\mathcal{G}})$$

$\hat{\mathcal{G}}$ is an *undirected* graph

Linear Agreement

Theorem

The consensus protocol over a weighted digraph converges to the average of the initial conditions for almost all initial conditions if and only if the digraph is balanced and weakly connected.

proof