Analysis and Control of Multi-Agent Systems

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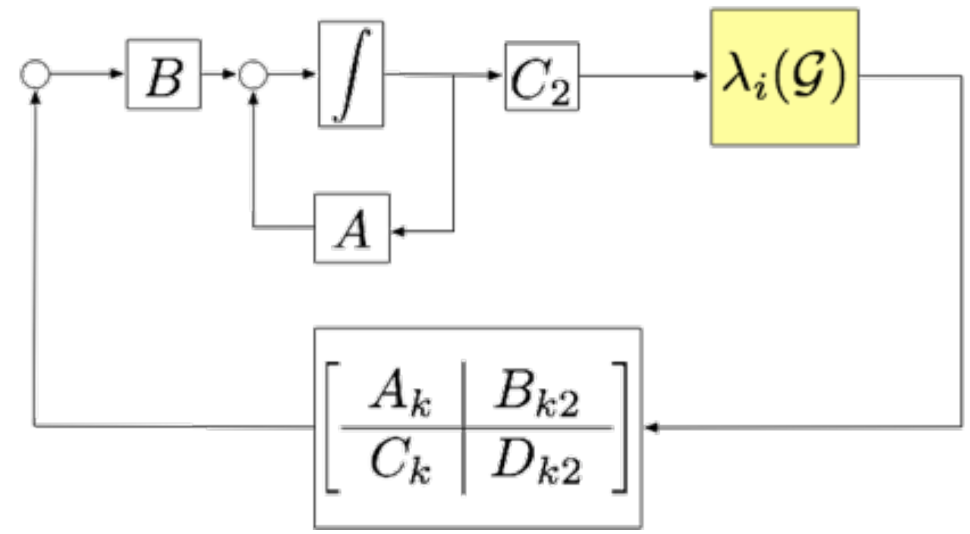
Control of Networks

Controlled Agreement

last time…

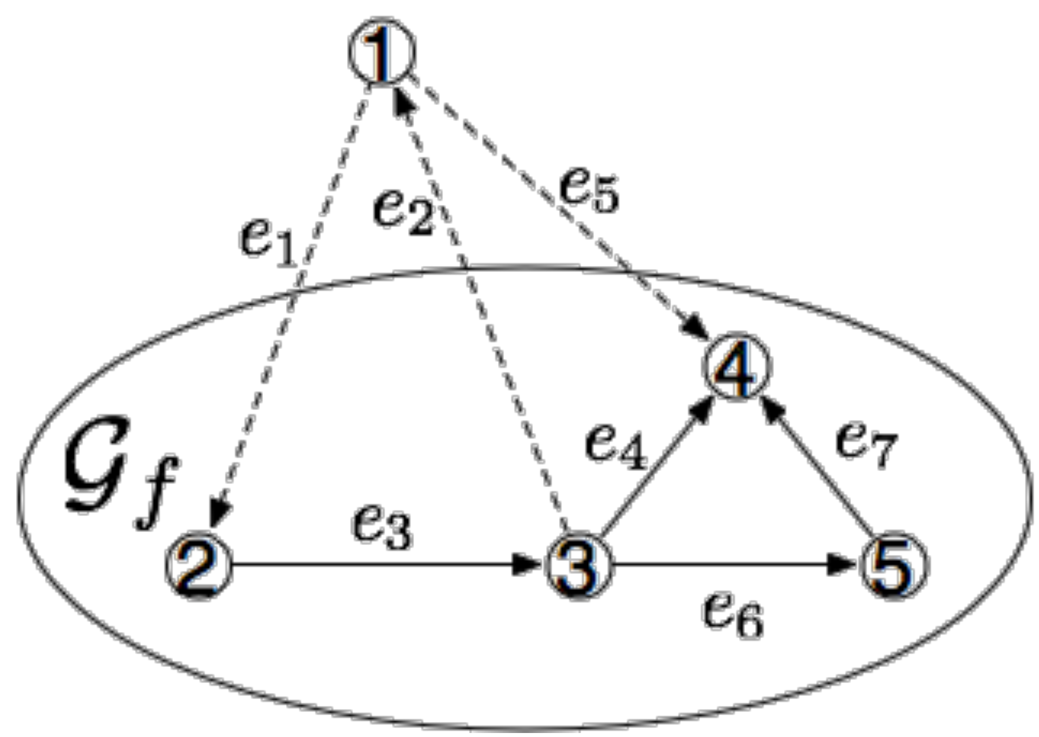
Formation Stabilization

- more general linear dynamics
- "consensus" feedback
- graph induced robustness margins
- normalized graph Laplacian

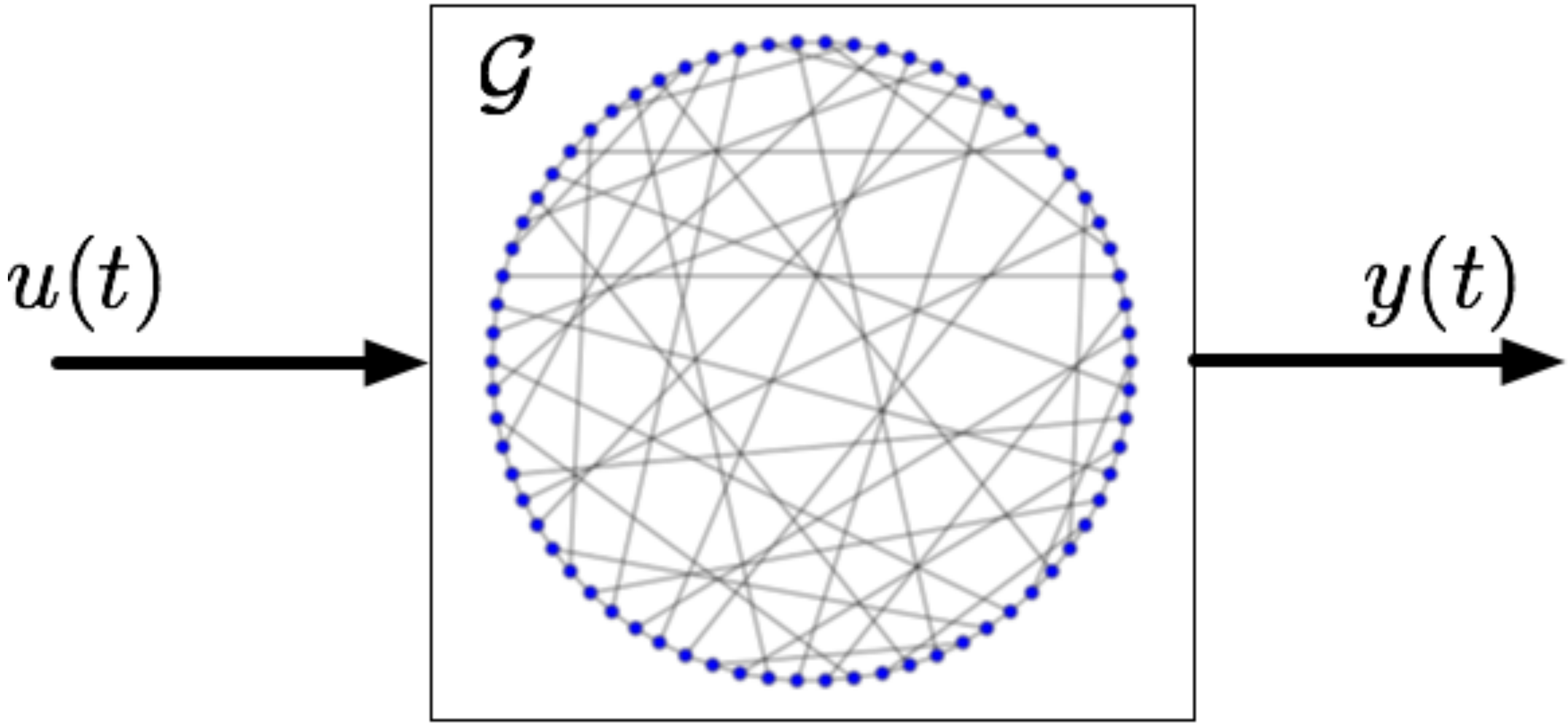


Controlled Agreement

- consensus protocol with a "rebel"
- input-output setup
- controllability



Networks as Systems

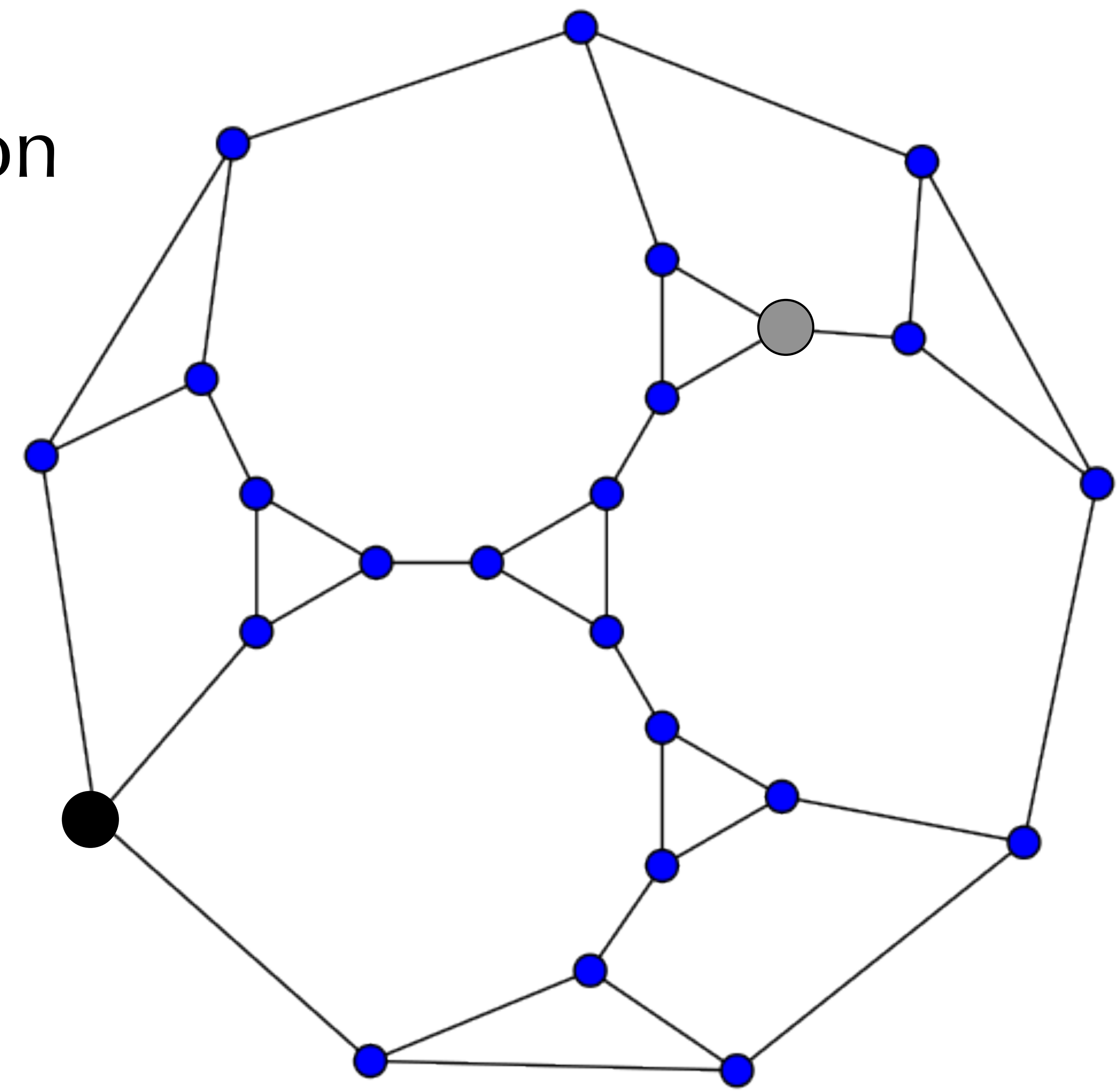


Can we relate system-theoretic properties to graph-theoretic concepts?

Networks as Systems

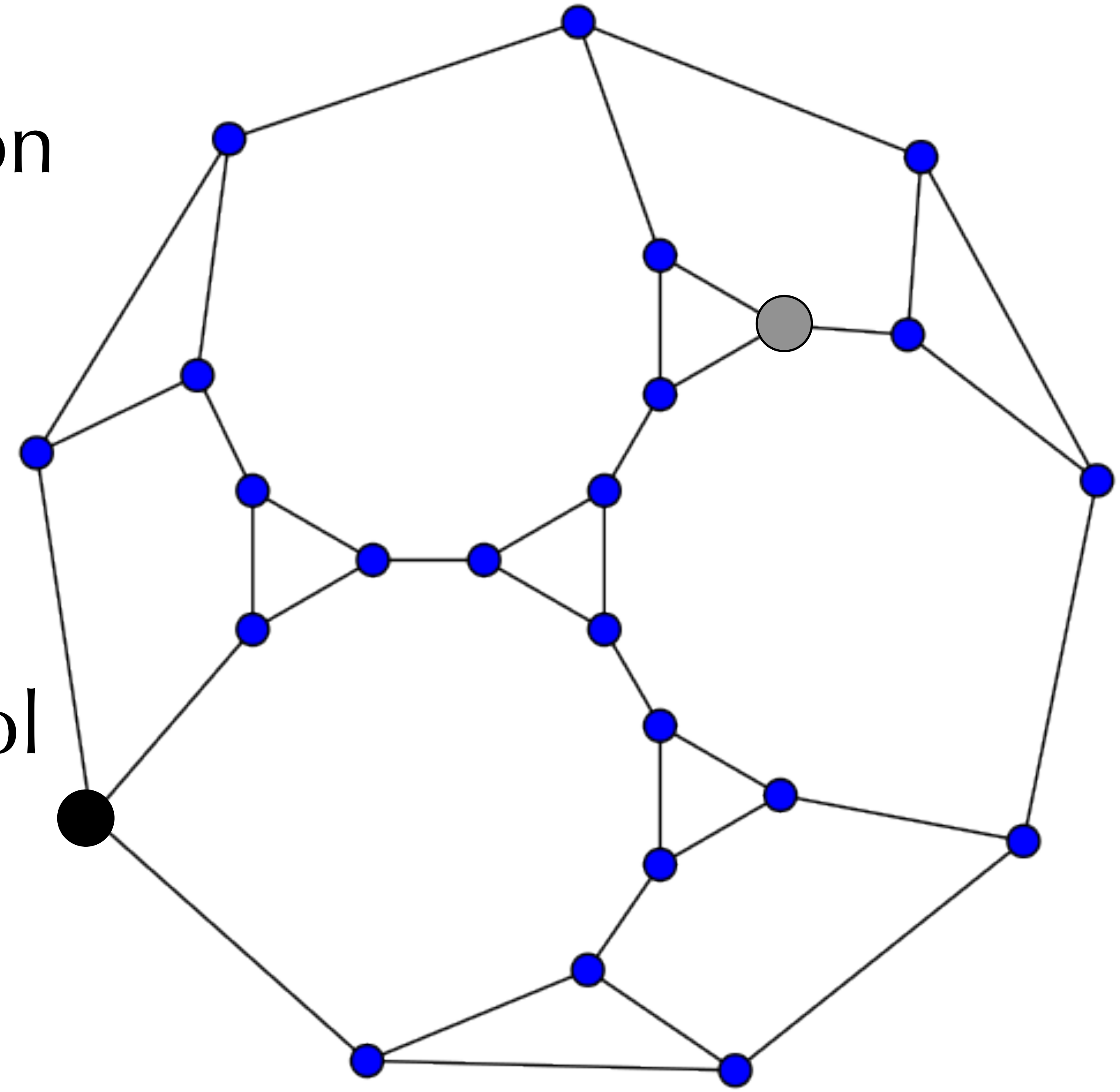
an 'input-output' modification of consensus networks

- attach to the network a
  - a *control node* ●
  - an *observation node* ●
- all other agents run consensus



Networks as Systems

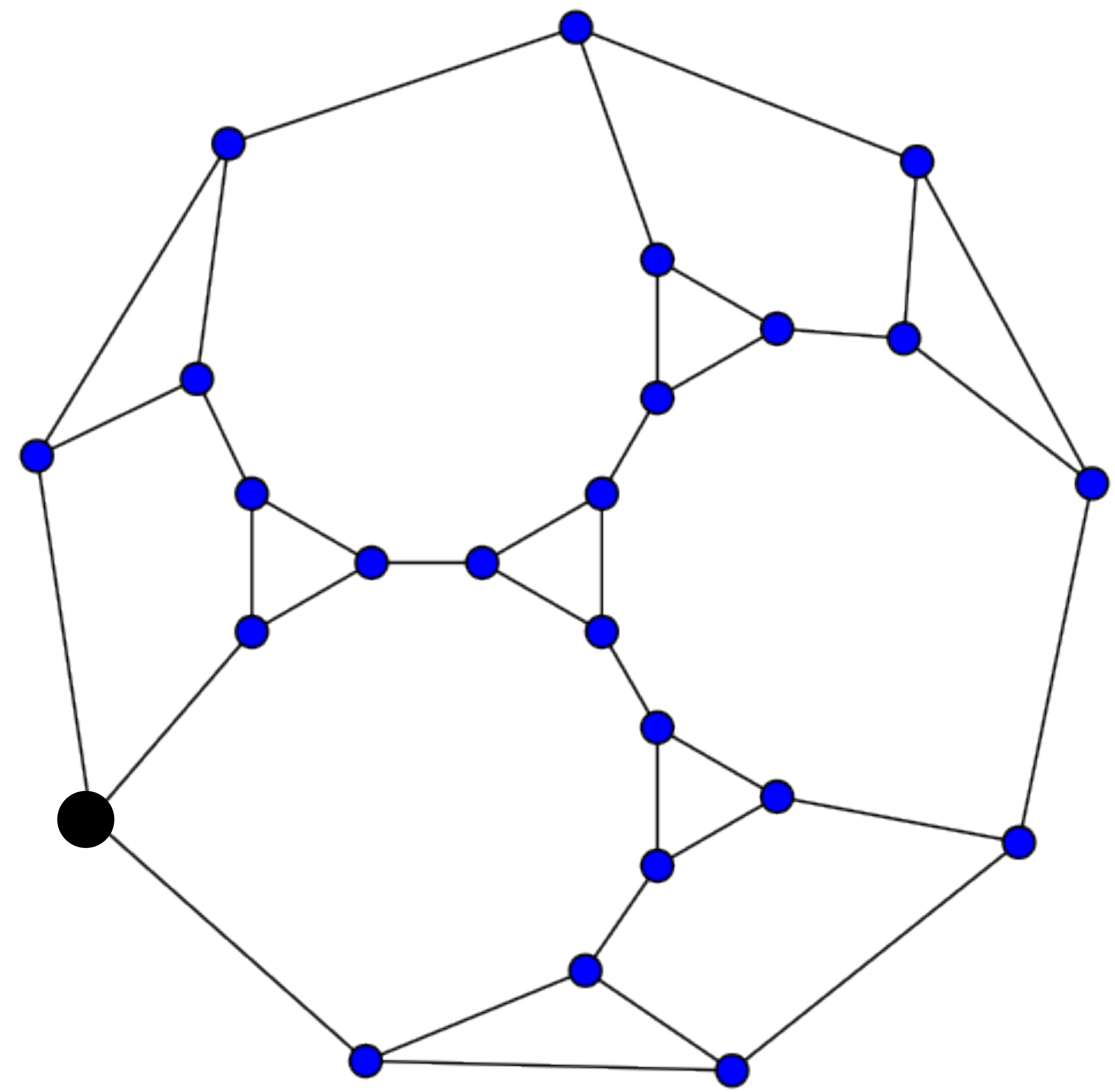
an 'input-output' modification of consensus networks



- can we *infiltrate* or *manipulate* the network behavior using these control nodes?
- can we *identify* properties of the network from the observation nodes?

Controlled Agreement

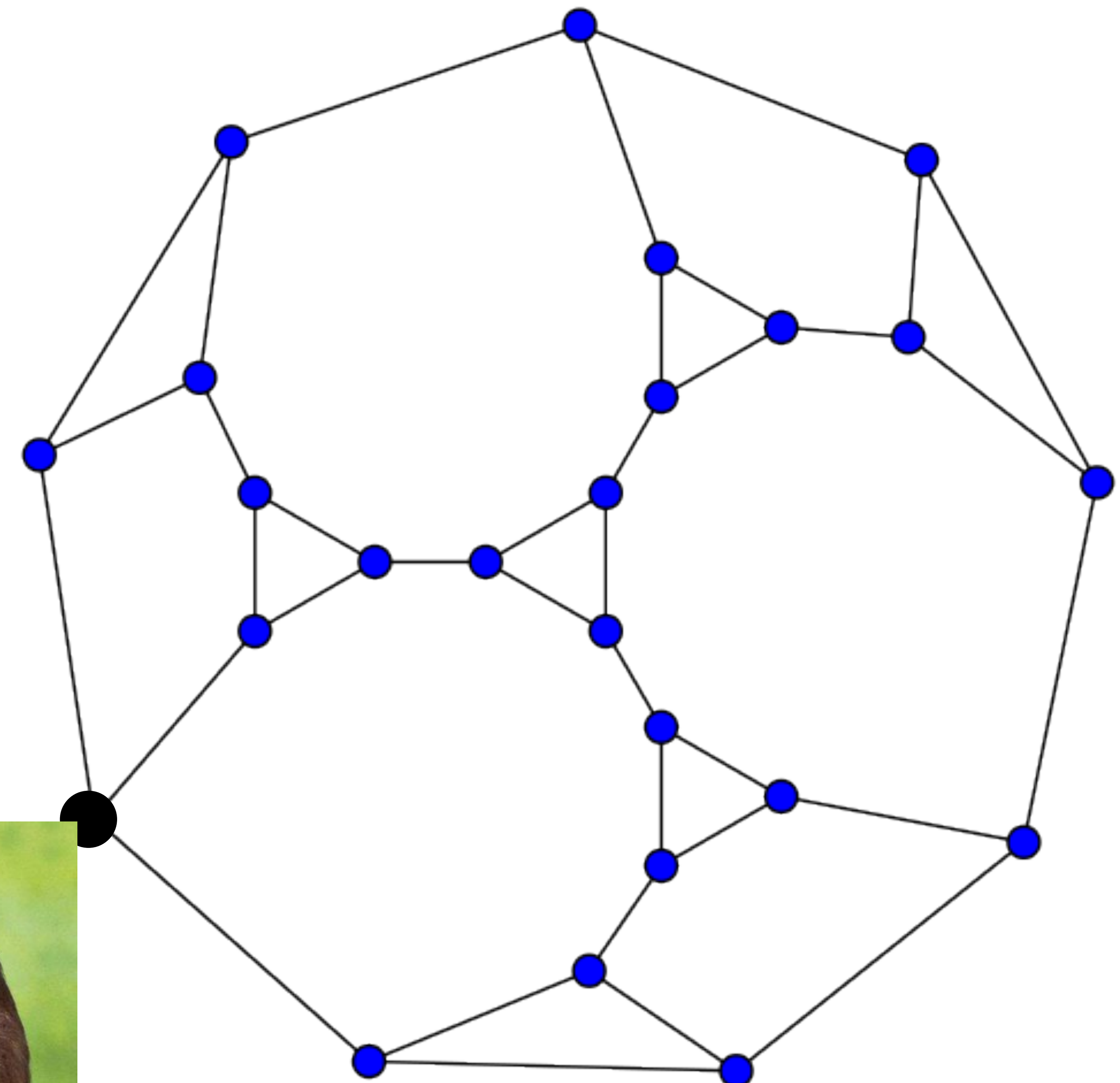
Is this system "controllable"?



Controlled Agreement

$$x(t) = -L(\mathcal{G})x(t)$$

assume one agent
“ignores” the protocol and
injects a different signal



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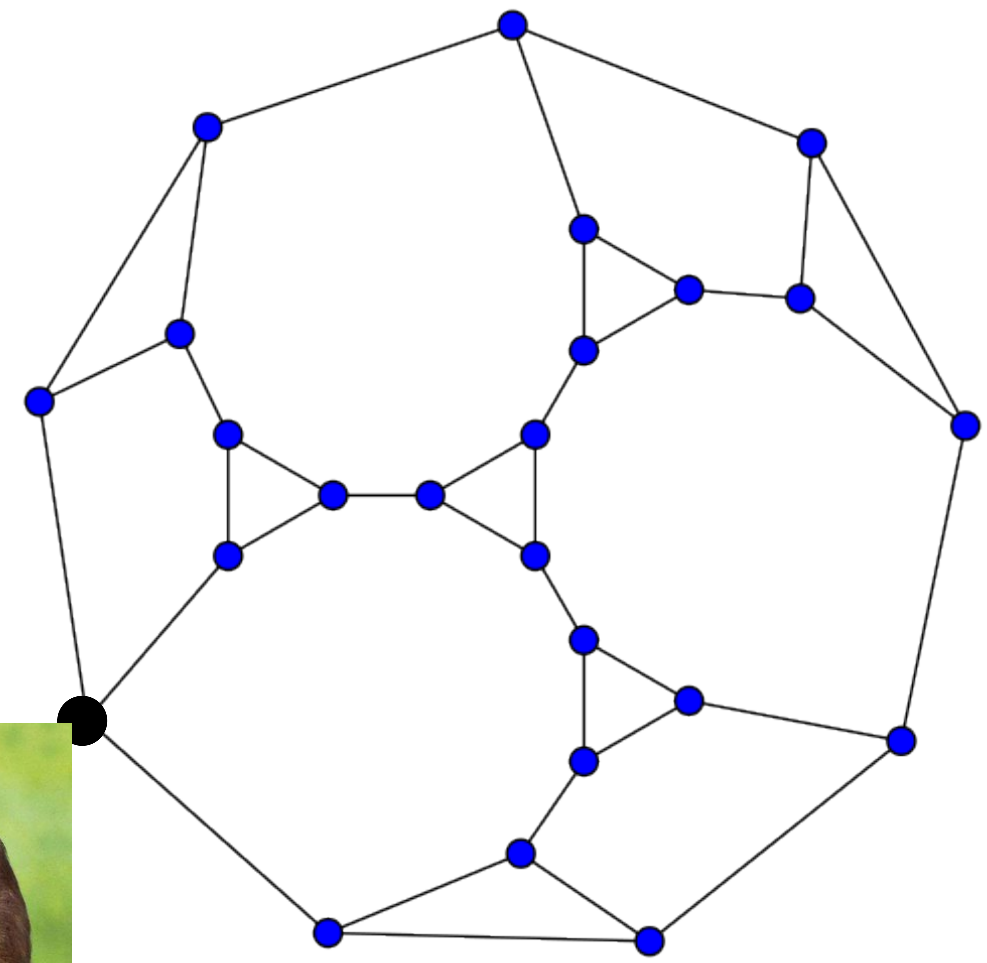
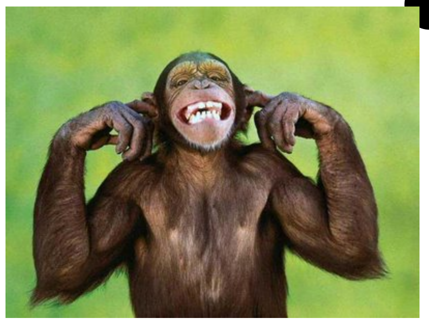


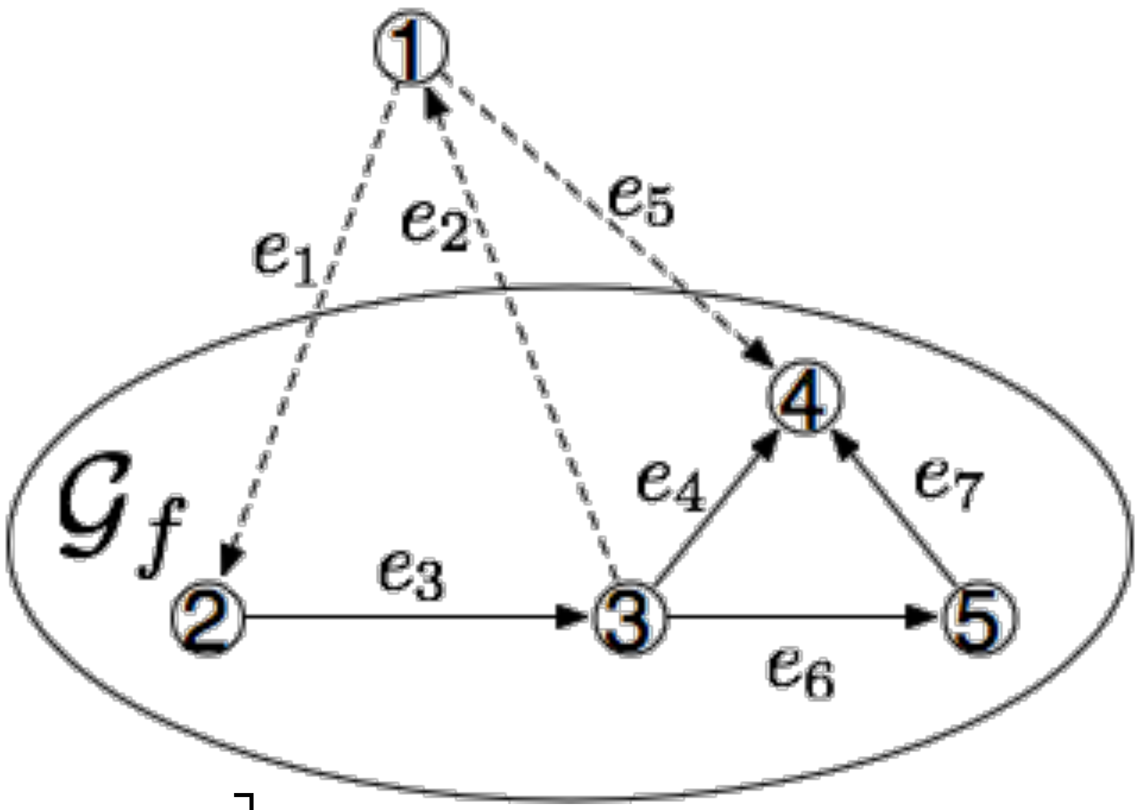


an input-output representation

$$\dot{x}_f(t) = A_f(\mathcal{G})x_f(t) + B_f(\mathcal{G})u(t)$$
$$y(t) = C_f(\mathcal{G})x_f(t)$$

Controlled Agreement

assume nodes are labeled so control node is node #1



$$E(\mathcal{G}) = \begin{bmatrix} e_1(\mathcal{G}) \\ E_f(\mathcal{G}) \end{bmatrix}$$

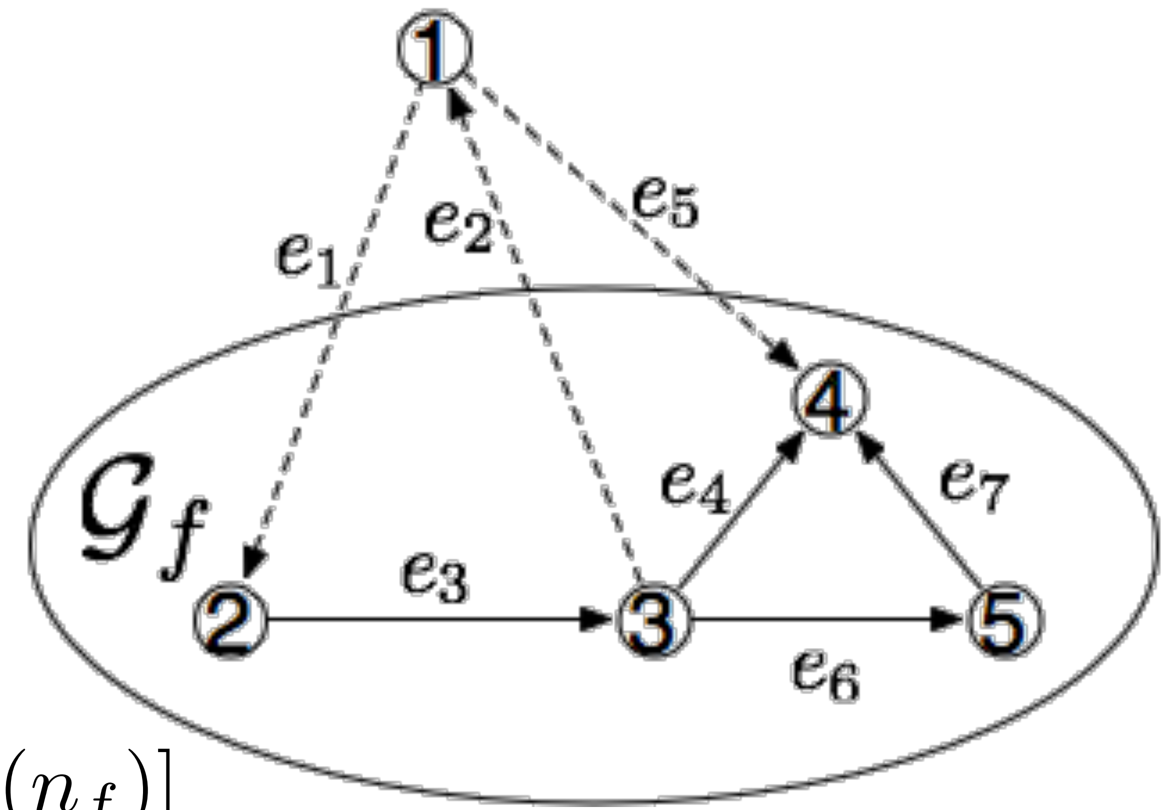
$$e_1(\mathcal{G}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$E_f(\mathcal{G}) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Controlled Agreement

rewrite the Laplacian…

$$L(\mathcal{G}) = \left[\begin{array}{c|c} e_1 e_1^T & e_1 E_f^T \\ \hline E_f e_1^T & E_f E_f^T \end{array}\right]$$



observe…

$$L(\mathcal{G}_f) = E_f E_f^T - \mathbf{diag}[d_i(1), \ldots, d_i(n_f)]$$

# of anchor nodes connected to follower node

Δ_f *input-to-state degree matrix*

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Controlled Agreement

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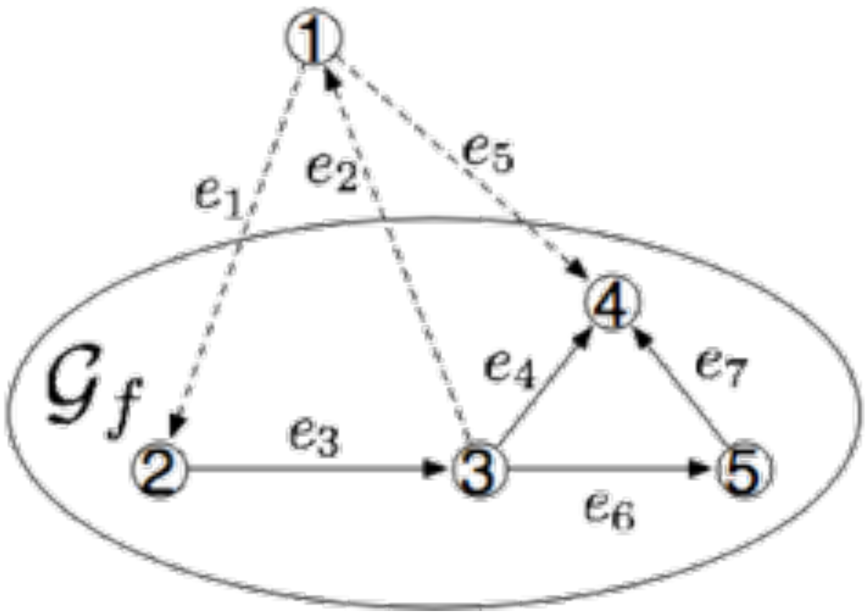


Input Indicator function for follower graph

$$\delta_1 = \begin{cases} 1, & v_i \sim v_1,\ v_i \in \mathcal{G}_f \\ 0, & o.w. \end{cases}$$

ex. $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

observe…

$$\delta_1 := -E_f e_1^T$$

"indicator" showing nodes in follower graph that are connected to anchor

Controlled Agreement

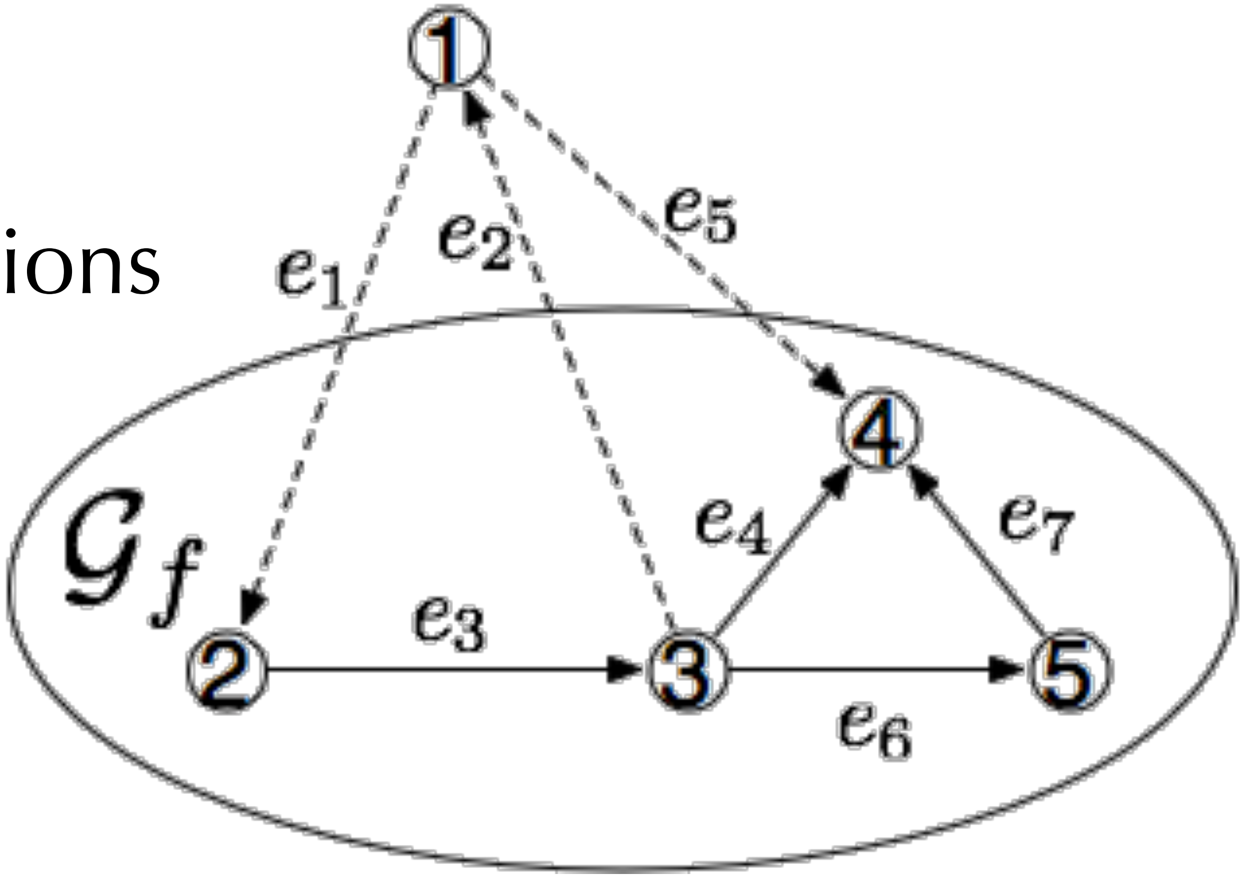
"Controlled Consensus"

$$\dot{x}_f(t) = -(L(\mathcal{G}_f) + \Delta_f)x_f(t) - \delta_1 x_1(t)$$

node 1 ignores everyone
follower nodes are "driven" by node 1

our control

Under what graph-theoretic conditions is this system uncontrollable?



Controllability

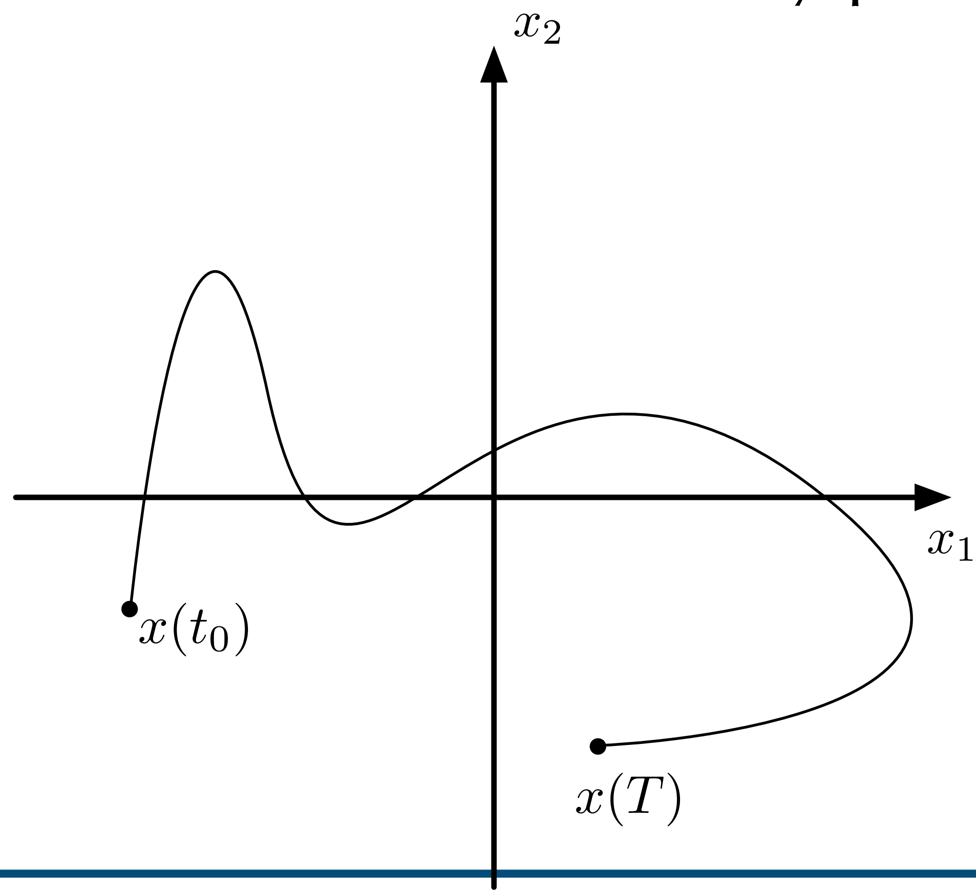
Consider a linear and time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(t) \in \mathbb{R}^n$$

$$u(t) \in \mathbb{R}^m$$

Does there exist a control *u(t)* that can steer the system state from an *arbitrary initial condition* to an *arbitrary* point in *finite time*?



Formation Stabilization

Theorem

The pair (A, B) is *controllable* if and only if

$$\mathbf{rk}\mathcal{C} = \mathbf{rk}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

Theorem

The pair (A, B) is *controllable* if and only if there is no left-eigenvector of A that is orthogonal to B, i.e.,

$$v^T B \neq 0, \forall v \neq 0, \ s.t. \ v^T A = \lambda v^T.$$

Controlled Agreement

Proposition

Given a single input linear system with symmetric state matrix A, if there exists an eigenvalue with geometric multiplicity greater than 1, then the system is uncontrollable.

Controlled Agreement

Lemma

The controlled consensus system is controllable if and only if $L(\mathcal{G})$ and $L(\mathcal{G}_f) + \Delta_f$ do not share an eigenvalue.

proof

assume uncontrollable: $\exists\, v \; s.t. \; (L(\mathcal{G}_f) + \Delta_f)v = \lambda v$

$$B_f^T v = 0$$

$$\begin{bmatrix} d_1 & B_f^T \\ B_f & L(\mathcal{G}_f) + \Delta_f \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v \end{bmatrix}$$

$\Rightarrow \lambda$ is an eigenvalue of $L(\mathcal{G})$

Controlled Agreement

Lemma

The controlled consensus system is controllable if and only if $L(\mathcal{G})$ and $L(\mathcal{G}_f)+\Delta_f$ do not share an eigenvalue.

proof

assume common eigenvalue $L(\mathcal{G})v=\lambda v,\ (L(\mathcal{G}_f)+\Delta_f)u=\lambda u$

$$\begin{bmatrix} d_1 & B_f^T \\ B_f & L(\mathcal{G}_f)+\Delta_f \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d_1 v_1 + B_f^T v_2 \\ B_f v_1 + (L(\mathcal{G}_f)+\Delta_f)v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \Rightarrow v_1 = 0, B_f^T v_2 = 0, v_2 = u$$

Controlled Agreement

observe...

$$(L(\mathcal{G}_f) + \Delta_f)\mathbf{1} = \delta_1$$

Corollary

The controlled consensus system is controllable if and only if none of the eigenvectors of $L(\mathcal{G}_f) + \Delta_f$ are orthogonal to $\mathbb{1}$.

Controlled Agreement

Corollary

If the single-input controlled agreement protocol is uncontrollable, then there exists an eigenvector v of A_f such that

$$\sum_{i \sim 1} v_i = 0.$$

proof

$$\text{uncontrollable} \Leftrightarrow \exists v \; s.t. \; A_f v = \lambda v, \; v^T \mathbf{1} = 0$$

$$\mathbf{1}^T (L(\mathcal{G}_f) + \Delta_f) v = \mathbf{1}^T \Delta_f v = 0 \; \Rightarrow \sum_{i \sim 1} v_i = 0$$

Controlled Agreement

Corollary

If the single-input controlled agreement protocol is uncontrollable, then there exists an eigenvector v of $L(\mathcal{G})$ that has a zero component at the index corresponding to the leader node (i.e., $v_1 = 0$).

proof

assume $A_f v = \lambda v, \ \mathbf{1}^T v = 0$

$$\begin{bmatrix} d_1 & B_f^T \\ B_f & A_f \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} B_f^T v \\ \lambda v \end{bmatrix}$$

uncontrollable means $B_f^T v = 0 \Rightarrow \begin{bmatrix} 0 \\ v \end{bmatrix}$ is an eigenvector with zero in component at index corresponding to anchor node!

From Algebraic to Graph Theoretic Conditions

"Controlled Consensus"

$$\dot{x}_f(t) = -(L(\mathcal{G}_f) + \Delta_f)x_f(t) - \delta_1 x_1(t)$$

all controllability results have been based on *algebraic tests*

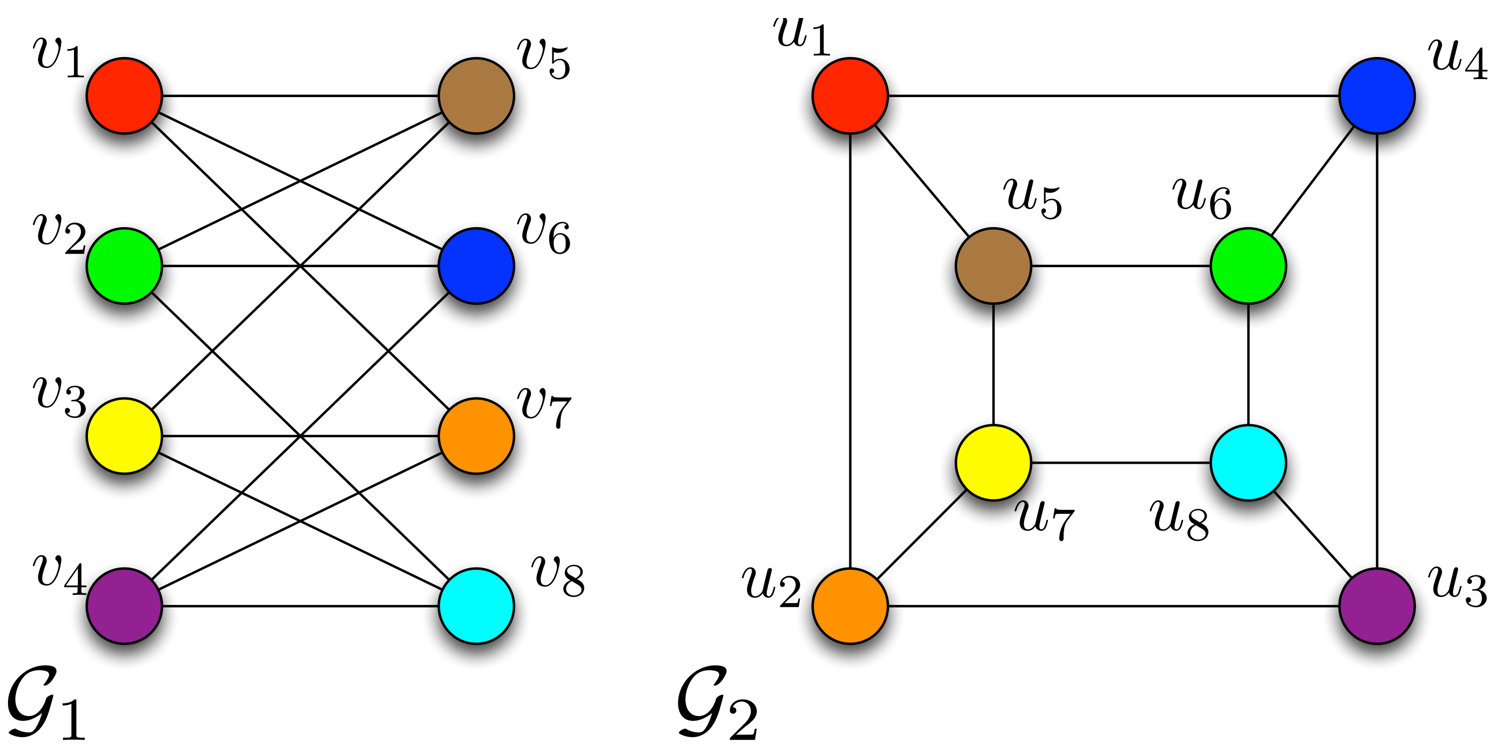
is there a *graph theoretic interpretation?*

Graph Symmetry and Graph Automorphisms

Graph Symmetry

Definition

Two graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are said to be *isomorphic* if there exists a bijection $\beta : \mathcal{V}_1 \to \mathcal{V}_2$ such that $(v_1, v_2) \in \mathcal{E}_1$ if and only if $(\beta(v_1), \beta(v_2)) \in \mathcal{E}_2$.



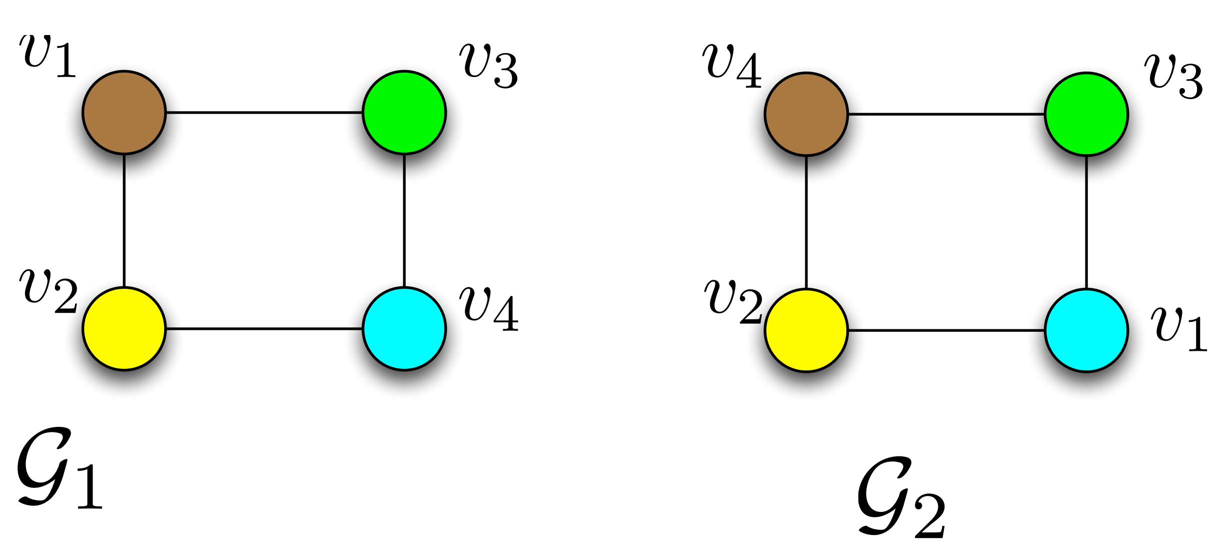
$$\beta(v_1) = u_1 \quad \beta(v_2) = u_6$$
$$\beta(v_3) = u_7 \quad \beta(v_4) = u_3$$
$$\beta(v_5) = u_5 \quad \beta(v_6) = u_4$$
$$\beta(v_7) = u_2 \quad \beta(v_8) = u_8$$

Graph Symmetry

Definition

An *automorphism* of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of its vertex set such that

$$\{\psi(v_i), \psi(v_j)\} \in \mathcal{E} \Leftrightarrow \{v_i, v_j\} \in \mathcal{E}.$$



$$\psi(v_1) = v_4 \quad \psi(v_2) = v_2$$
$$\psi(v_3) = v_3 \quad \psi(v_4) = v_1$$

an automorphism is an isomorphism of a graph “onto itself”

Graph Symmetry

Proposition

Let $A(\mathcal{G})$ be the adjacency matrix of the graph $\mathcal{G}$ and ψ a permutation on its vertex set $\mathcal{V}$. Associate with this permutation the permutation matrix Ψ such that

$$[\Psi]_{ij} = \begin{cases} 1, & if\, \psi(i) = j, \\ 0, & o.w. \end{cases} .$$

Then ψ is an automorphism of $\mathcal{G}$ if and only if

$$\Psi A(\mathcal{G}) = A(\mathcal{G})\Psi$$

Graph Symmetry

Definition

The controlled agreement system is *input symmetric* with respect to the anchor node if there exists a nonidentity permutation matrix J such that

$$JA_f = A_f J.$$

$$JA_f = A_f J$$

$$J(L(\mathcal{G}_f) + \Delta_f) = (L(\mathcal{G}_f) + \Delta_f)J$$

$$J(\Delta(\mathcal{G}_f) - A(\mathcal{G}_f) + \Delta_f) = (\Delta(\mathcal{G}_f) - A(\mathcal{G}_f) + \Delta_f)J$$

$$J\underbrace{(\Delta(\mathcal{G}_f) + \Delta_f)}_{\tilde{\Delta}} - JA(\mathcal{G}_f) = \tilde{\Delta}J - A(\mathcal{G}_f)J$$

Graph Symmetry

Proposition

Let Ψ be the matrix associated with a permutation ψ. Then

$$\Psi(\Delta(\mathcal{G}_f) + \Delta_f) = (\Delta(\mathcal{G}_f) + \Delta_f)\Psi$$

if and only if, for all i

$$d_i(\mathcal{G}_f) + \delta_1(i) = d_{\psi(i)}(\mathcal{G}_f) + \delta_1(\psi(i)).$$

In the case where ψ is an automorphism of $\mathcal{G}_f$, the condition becomes

$$\delta_1(i) = \delta_1(\psi(i)), \forall i.$$

recall: $\delta_1 = B_f = -E_f e_1^T$

Controlled Agreement and Symmetry

Proposition

The controlled agreement protocol is input symmetric if and only if there is a nonidentity automorphism for $\mathcal{G}_f$ such that the input indicator vector remains invariant under its action.

Corollary

The controlled agreement protocol is input asymmetric if the automorphism graph of $\mathcal{G}_f$ only contains the trivial (identity) permutation.

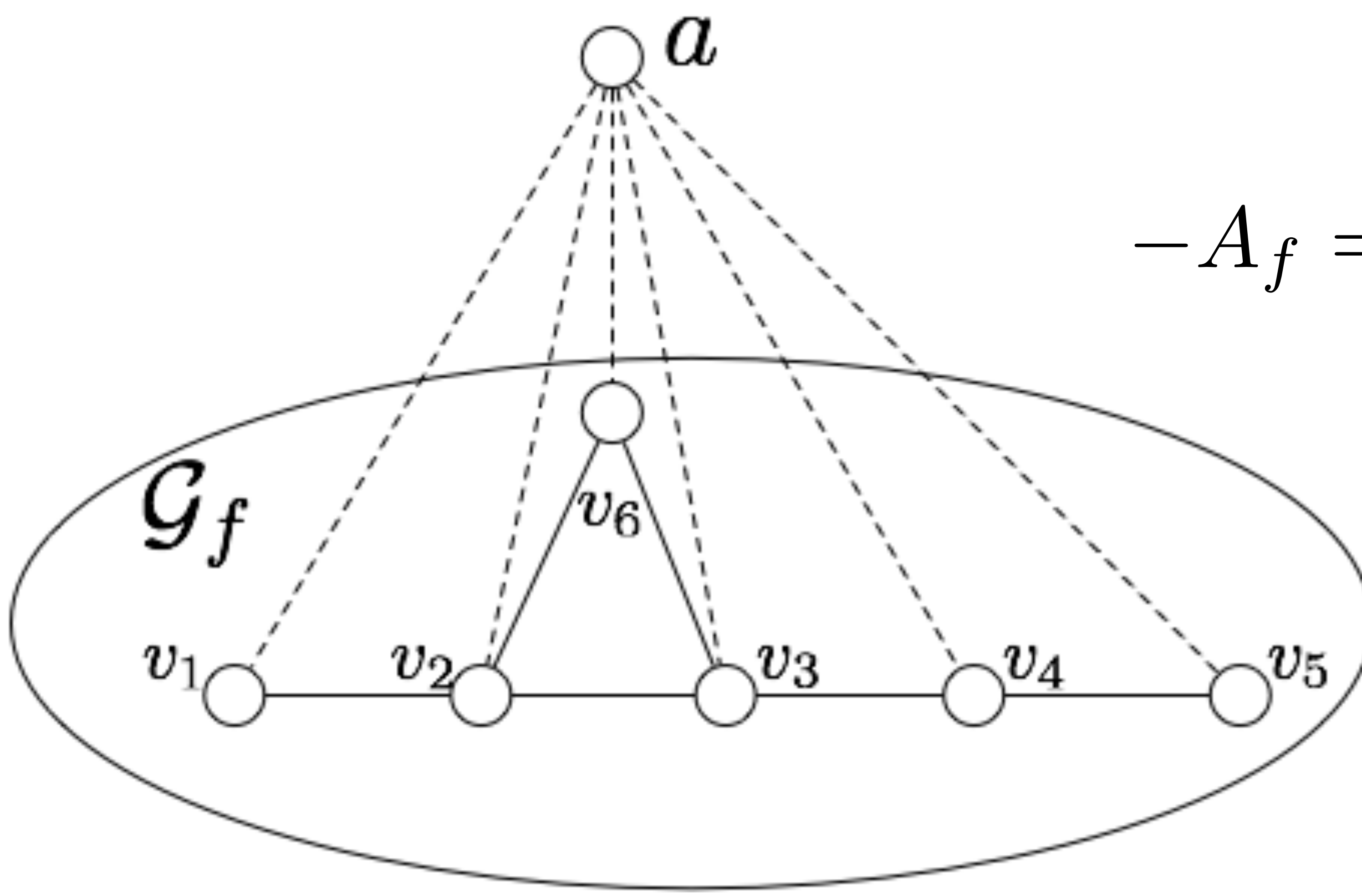
Controlled Agreement and Symmetry

Theorem

The controlled agreement protocol is uncontrollable if it is input symmetric. Equivalently, it is uncontrollable if $\mathcal{G}_f$ admits a nonidentity automorphism for which the input indicator vector remains invariant under its action.

Input symmetry is *not* a necessary condition for controllability of the controlled agreement protocol!

A Counter Example

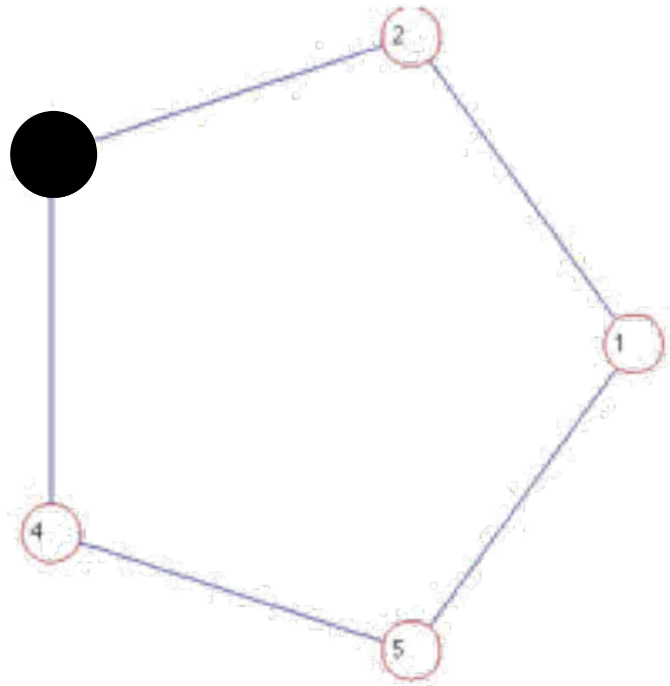


$$-A_f = L(\mathcal{G}_f) + I$$

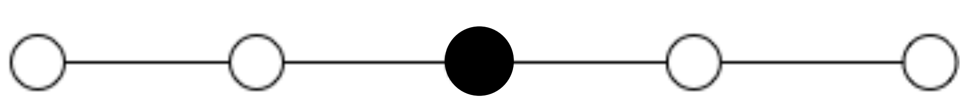
follower graph is the *smallest asymmetric graph*; it does not admit any nonidentity automorphism

corresponding system is *not* input symmetric with respect to node *a*, but controlled agreement is not controllable.

Cycle Graphs



the cycle graph is uncontrollable from any single anchor node!



the path graph with odd number of vertices is always uncontrollable from the center node