Analysis and Control of Multi-Agent Systems

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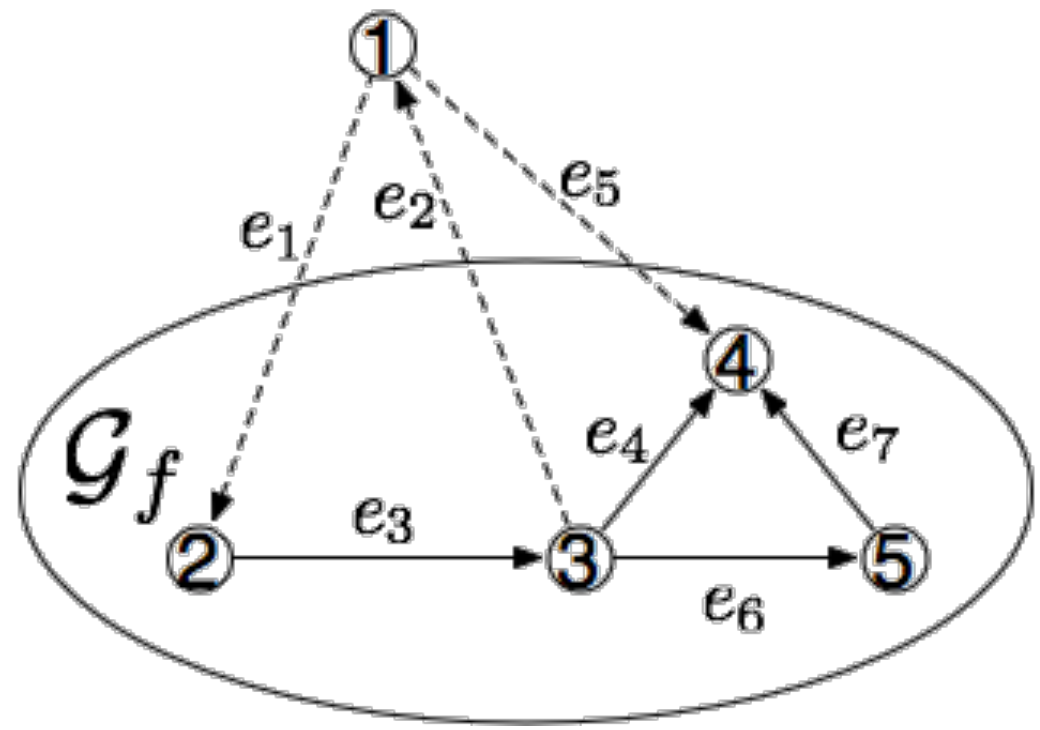
Control of Networks

Edge Agreement and Consensus Performance

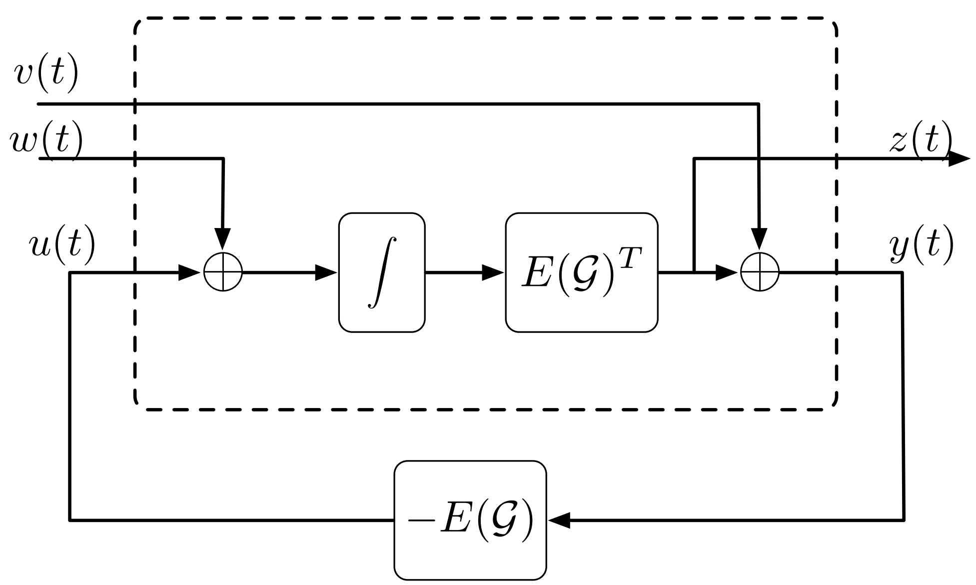
last time…

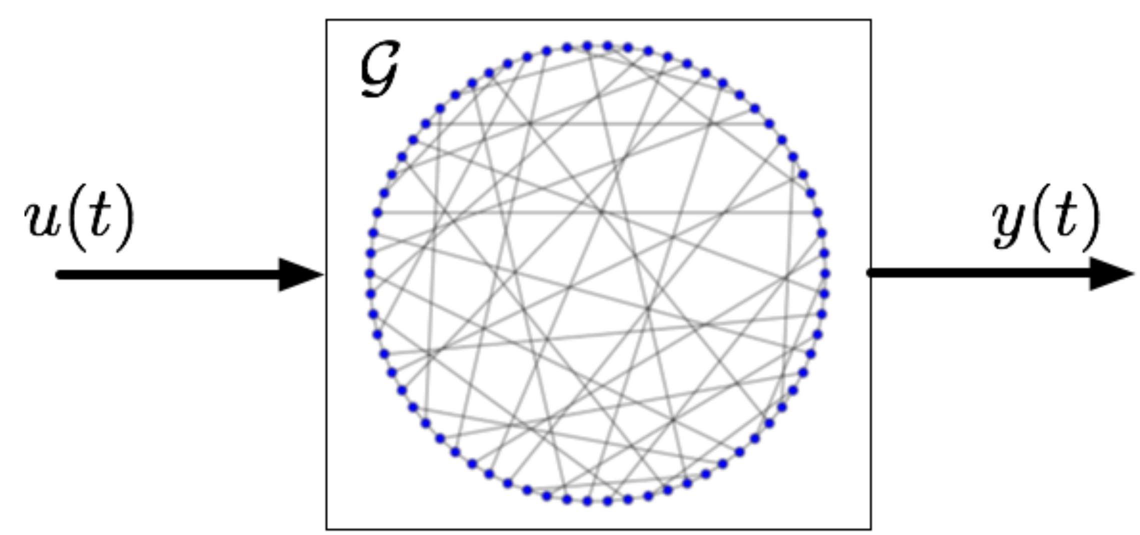
Controlled Agreement

- consensus protocol with a "rebel"
- input-output setup
- controllability



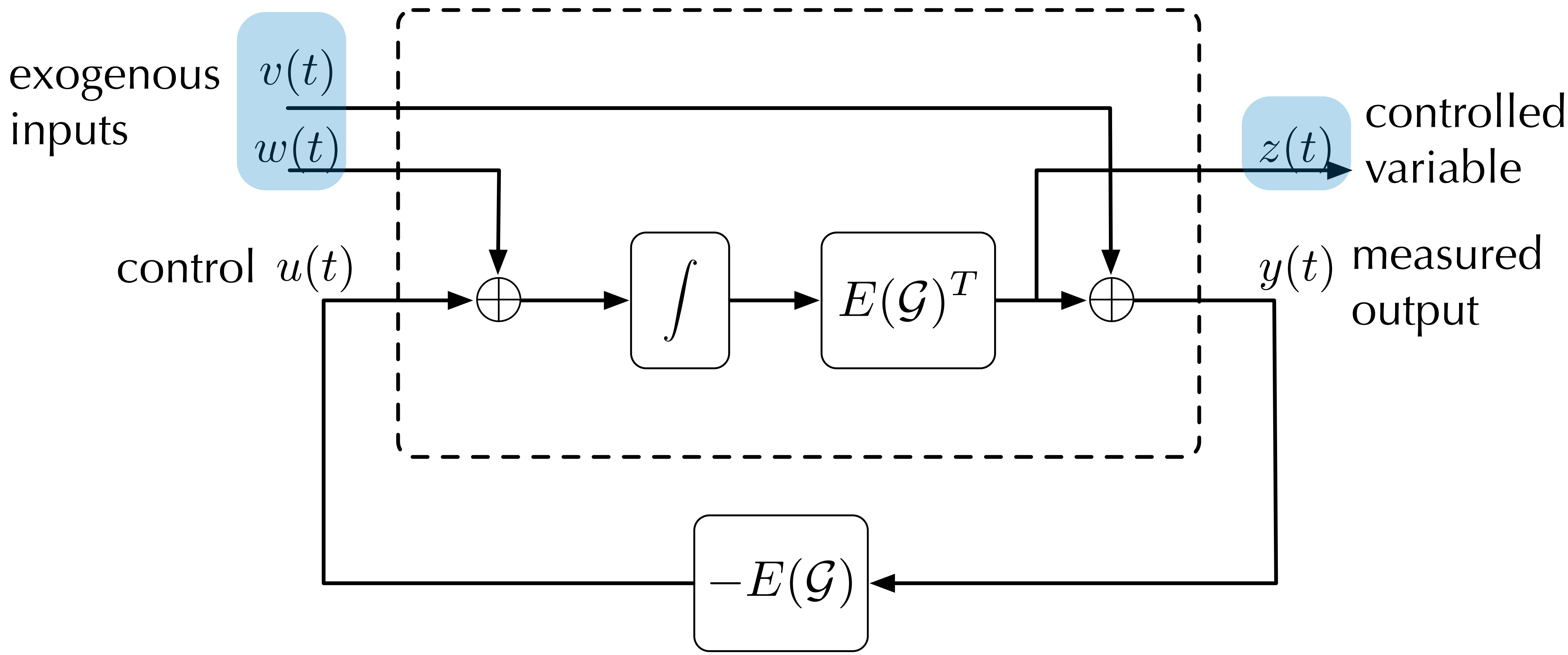
Performance of Consensus





Consensus with Exogenous Inputs

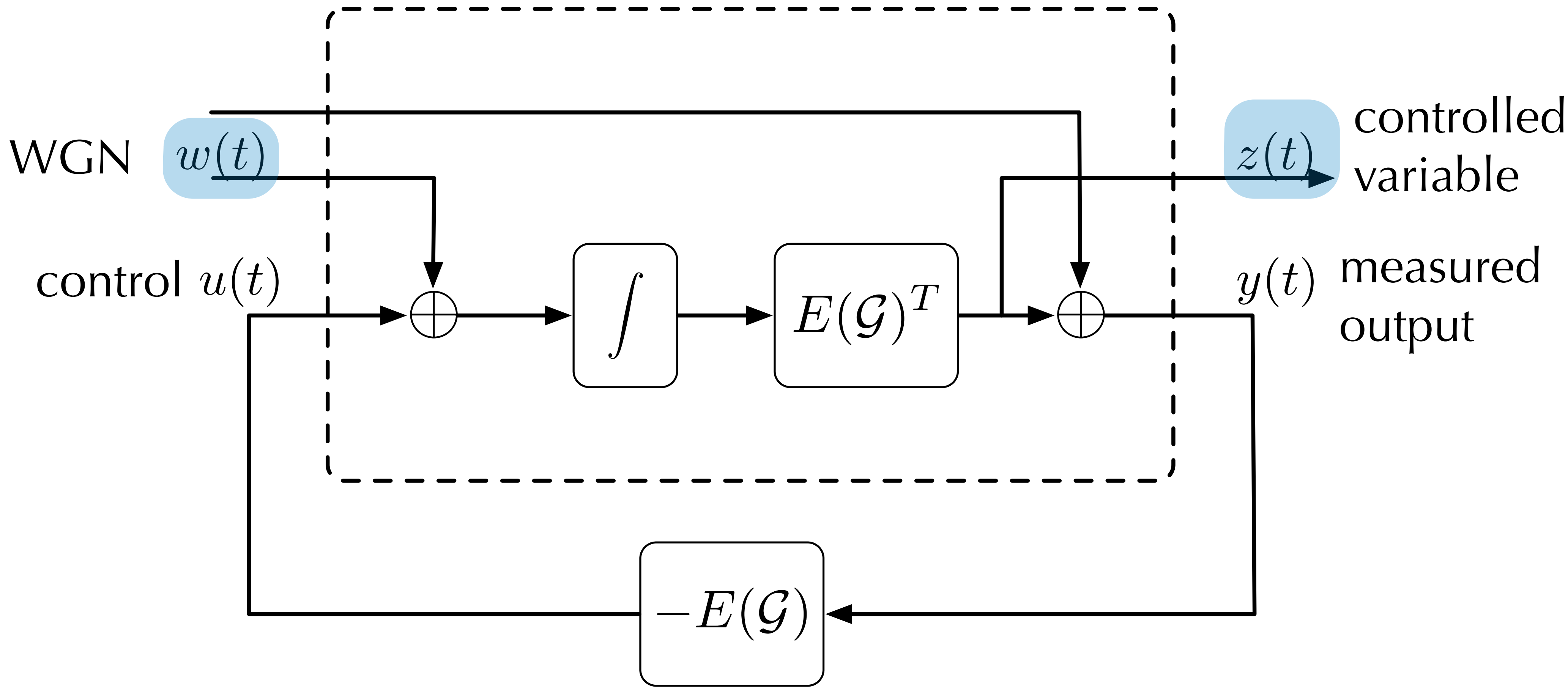
An 'input-output' consensus model



$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) = -L(\mathcal{G})x(t) + \begin{bmatrix} I & -E(\mathcal{G}) \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) = E(\mathcal{G})^T x(t). \end{cases}$$

Consensus with Exogenous Inputs

what happens when consensus is driven
by Gaussian white noise?



$$\begin{cases} \dot{x}(t) &= -L(\mathcal{G})x(t) + w(t) \\ z(t) &= E(\mathcal{G})^T x(t) \end{cases}$$

Consensus with Exogenous Inputs

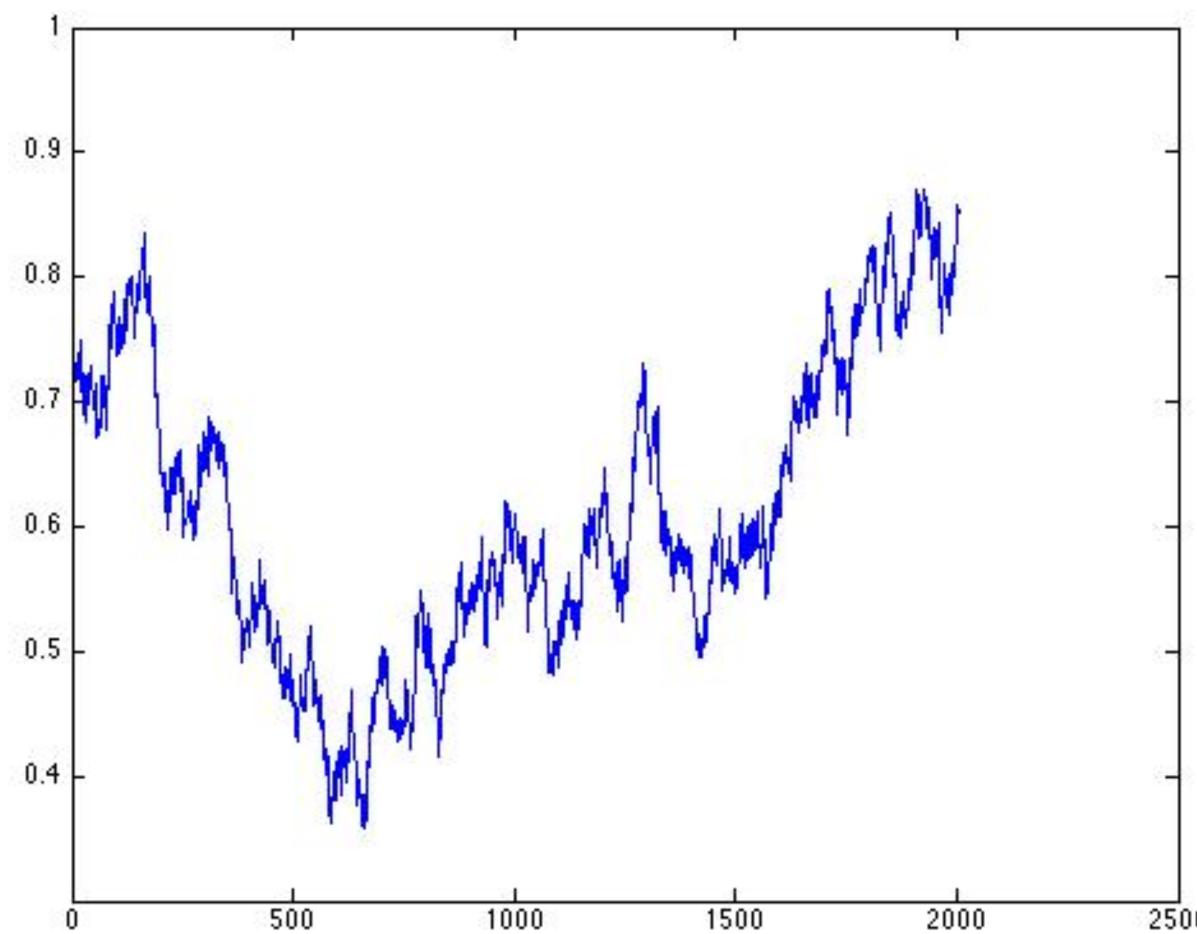
what happens when consensus is driven by Gaussian white noise?

$$\dot{\overline{x}}(t) = \frac{1}{n}\mathbf{1}^T w(t)$$

average is "driven" by noises…

$$\mathcal{E}(\overline{x}(t)^2) = \frac{\sigma_w^2}{n}t$$

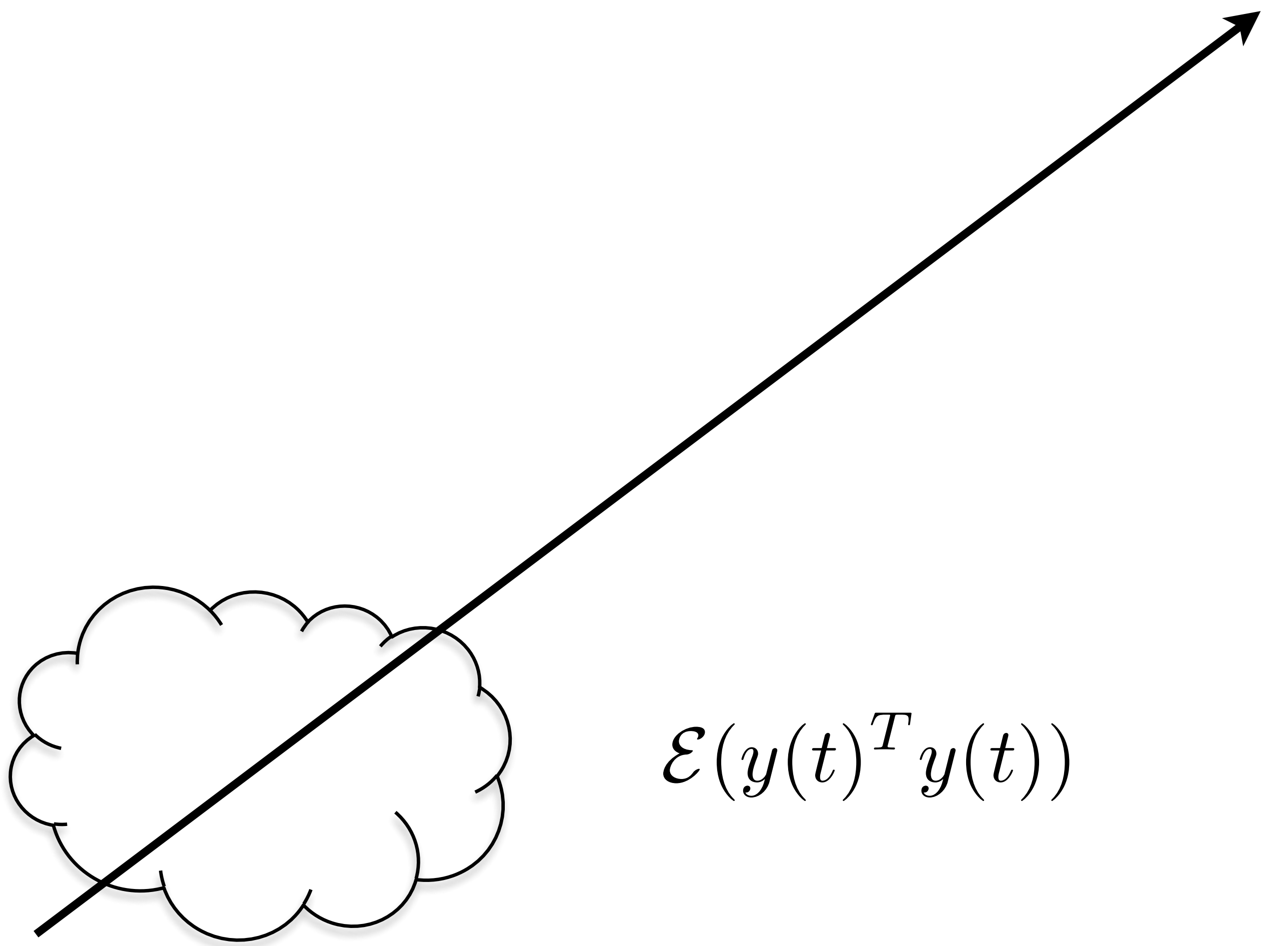
a *random walk*



Consensus with Exogenous Inputs

what happens when consensus is driven by Gaussian white noise?

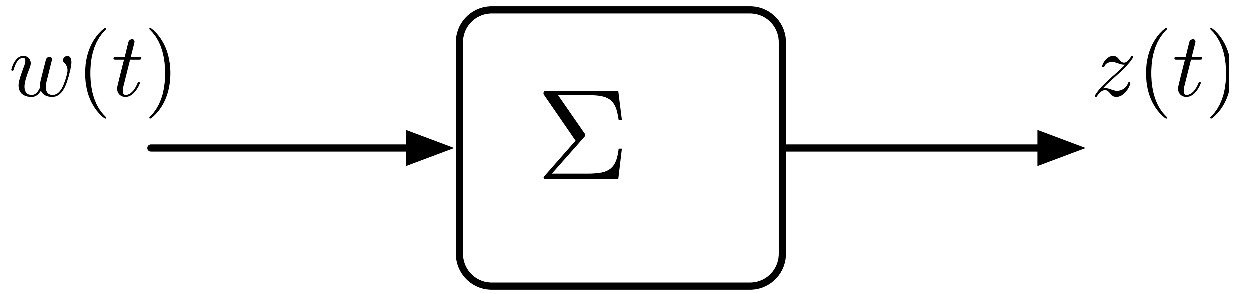
$$\mathcal{N}(E(\mathcal{G})^T) = \text{span}\{\mathbf{1}\}$$



$$\mathcal{E}(y(t)^T y(t))$$

When driven by noise, it is meaningful to examine how noises effect the stead-state covariance of the *relative states*

$\mathcal{H}_2$ Performance of Linear Systems



A stable linear system

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right.$$

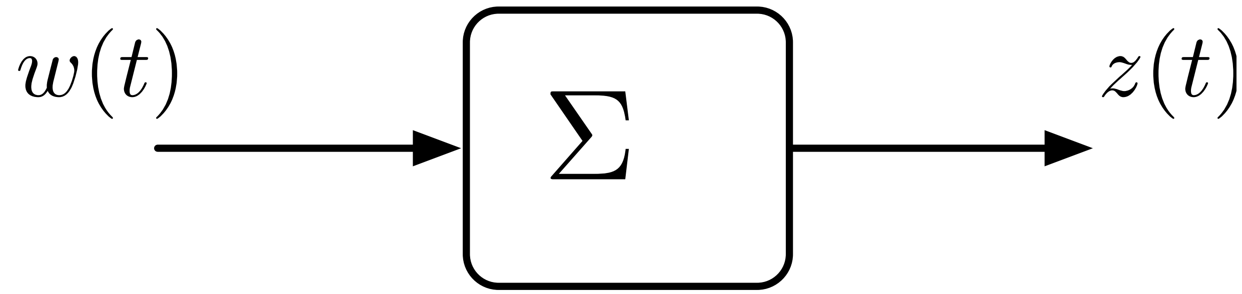
$\mathcal{H}_2$ System Norm

$$\|\Sigma\|_2 = \sqrt{\mathbf{trace}\, CPC^T}$$

Controllability Gramian

$$P = \int_0^\infty e^{At} BB^T e^{A^T t} dt \qquad AP + PA^T + BB^T = 0$$

$\mathcal{H}_2$ Performance of Linear Systems

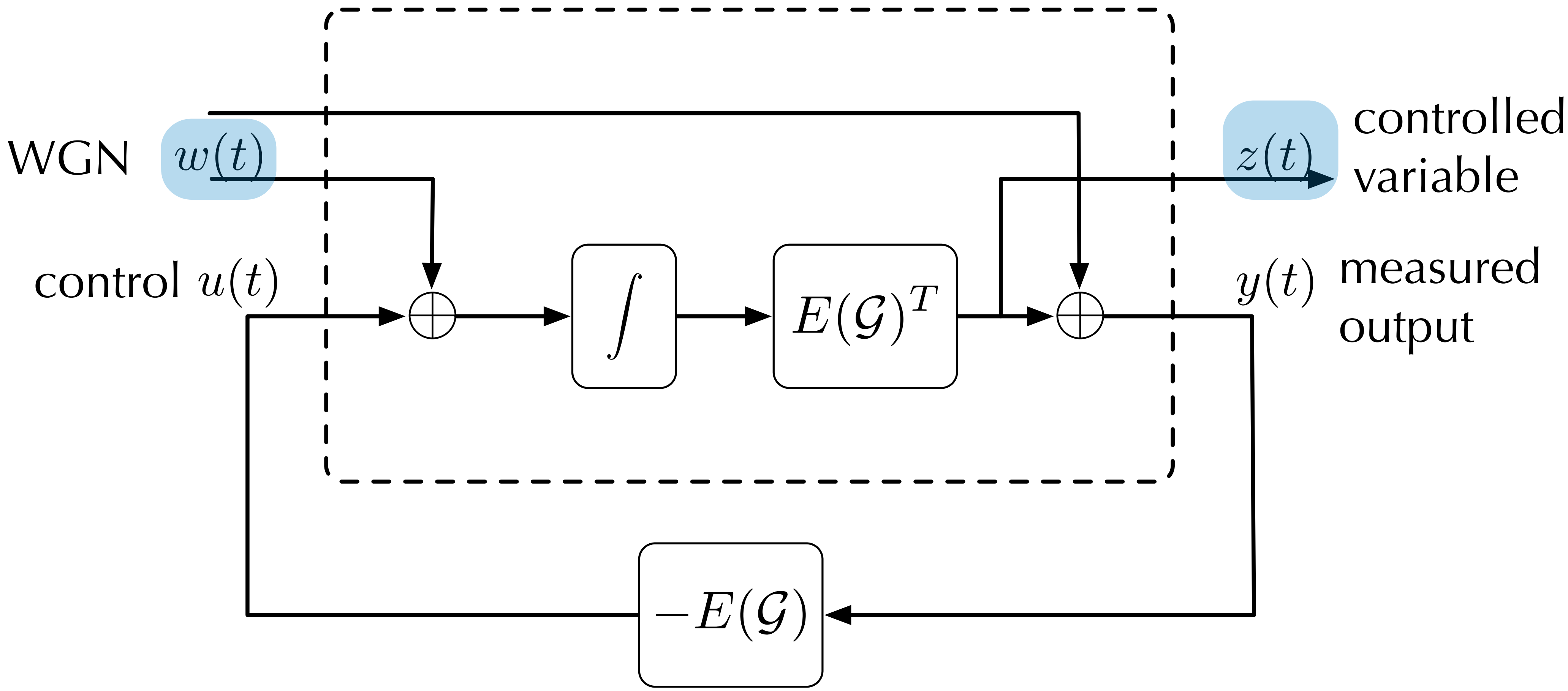


A stable linear system

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right.$$

for linear systems driven by white Gaussian noise, the $\mathcal{H}_2$ system norm can be interpreted as a *bound* on the *steady-state covariance* of the output

A Minimal Realization



$$S = \begin{bmatrix} P & \frac{1}{\sqrt{n}}\mathbb{1} \end{bmatrix}$$

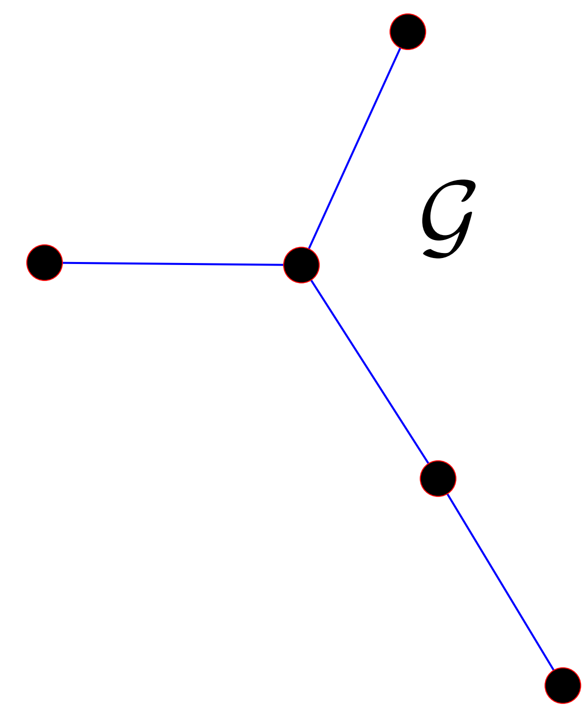
$$\mathbb{1}^T P = 0$$

$$\tilde{x}(t) = S^{-1}x(t)$$

$$\begin{cases} \dot{\tilde{x}}(t) &= -S^{-1}L(\mathcal{G})S\tilde{x}(t) + S^{-1}w(t) \\ z(t) &= E(\mathcal{G})^T S\tilde{x}(t) \end{cases}$$

A Minimal Realization

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}}(t) & = & -S^{-1}L(\mathcal{G})S\tilde{x}(t) + S^{-1}w(t) \\ z(t) & = & E(\mathcal{G})^T S\tilde{x}(t) \end{array} \right.$$

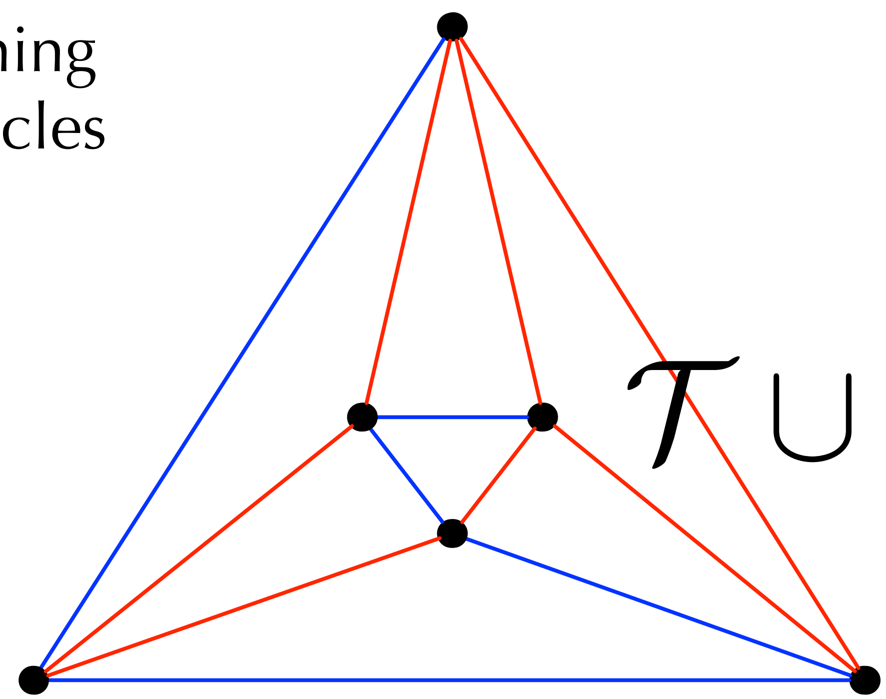


$$S = \left[\begin{array}{cccc|c} -0.45 & -0.45 & -0.45 & -0.45 & 0.45 \\ 0.86 & -0.14 & -0.14 & -0.14 & 0.45 \\ -0.14 & 0.86 & -0.14 & -0.14 & 0.45 \\ -0.14 & -0.14 & 0.86 & -0.14 & 0.45 \\ -0.14 & -0.14 & -0.14 & 0.86 & 0.45 \end{array} \right]$$

$$S^{-1}L(\mathcal{G})S = \left[\begin{array}{cccc|c} 2.90 & 0.90 & 0.90 & -0.40 & 0.00 \\ 0.90 & 1.90 & 0.90 & 0.60 & 0.00 \\ 0.90 & 0.90 & 1.90 & 0.60 & -0.00 \\ -0.40 & 0.60 & 0.60 & 1.29 & -0.00 \\ \hline 0.00 & 0.00 & 0.00 & 0.00 & -0.00 \end{array} \right]$$

Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles



$\mathcal{T} \cup \mathcal{C}$

a spanning tree

remaining edges "complete cycles"

$$E(\mathcal{G}) = E(\mathcal{T}) \underbrace{\left[\; I \quad T_{(\mathcal{T},\mathcal{C})} \; \right]}_{\mathcal{R}_{(\mathcal{T},\mathcal{C})}}$$

Cycles are a "linear combination" of edges in a spanning tree

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^T E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^T}_{E_{\mathcal{T}}^L} E(\mathcal{C})$$

$\mathcal{R}_{(\mathcal{T},\mathcal{C})}$ rows form a basis for the *cut space* of the graph

Edge Laplacian

$$\begin{cases} \dot{\tilde{x}}(t) & = & -S^{-1}L(\mathcal{G})S\tilde{x}(t) + S^{-1}w(t) \\ z(t) & = & E(\mathcal{G})^T S\tilde{x}(t) \end{cases}$$

$$\left[\begin{array}{c} x_{\mathcal{T}}(t) \\ \overline{x}(t) \end{array}\right] = \underbrace{\left[\begin{array}{c} E(\mathcal{T})^T \\ \frac{1}{n}\mathbb{1}^T \end{array}\right]}_{S^{-1}} x(t)$$

$$\left[\begin{array}{c} \dot{x}_{\mathcal{T}}(t) \\ \dot{\overline{x}}(t) \end{array}\right] = \left[\begin{array}{cc} \left(E(\mathcal{T})^T E(\mathcal{T})\right) \mathcal{R}_{(\mathcal{T},\mathcal{C})} \mathcal{R}^T_{(\mathcal{T},\mathcal{C})} & \mathbf{0} \\ \mathbf{0} & 0 \end{array}\right] x(t) \left[\begin{array}{c} x_{\mathcal{T}}(t) \\ \overline{x}(t) \end{array}\right] + \left[\begin{array}{c} E(\mathcal{T})^T \\ \frac{1}{n}\mathbb{1}^T \end{array}\right] w(t)$$

Edge Laplacian

Edge Laplacian

$$L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$$

Essential Edge Laplacian

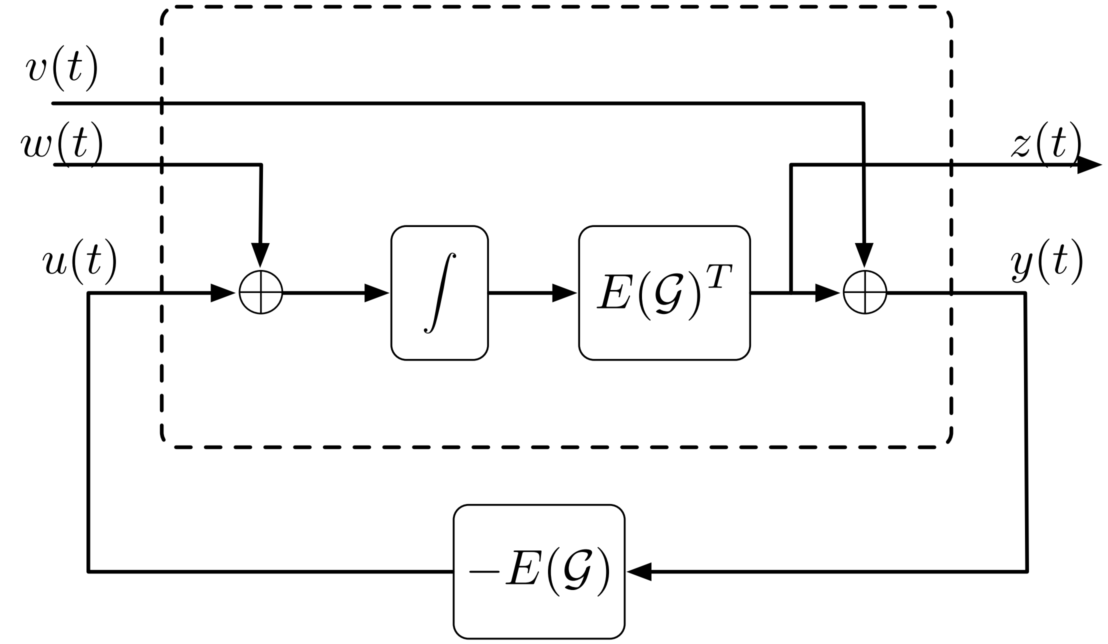
$$L_e(\mathcal{T}) \mathcal{R}_{(\mathcal{T},\mathcal{C})} \mathcal{R}_{(\mathcal{T},\mathcal{C})}^T$$

similarity between edge
and graph Laplacians

$L(\mathcal{G})$ $\quad$ $L_e(\mathcal{G})$

The Edge Agreement Problem

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) &= -L(\mathcal{G})x(t) + \begin{bmatrix} I & -E(\mathcal{G}) \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) &= E(\mathcal{G})^T x(t). \end{cases}$$

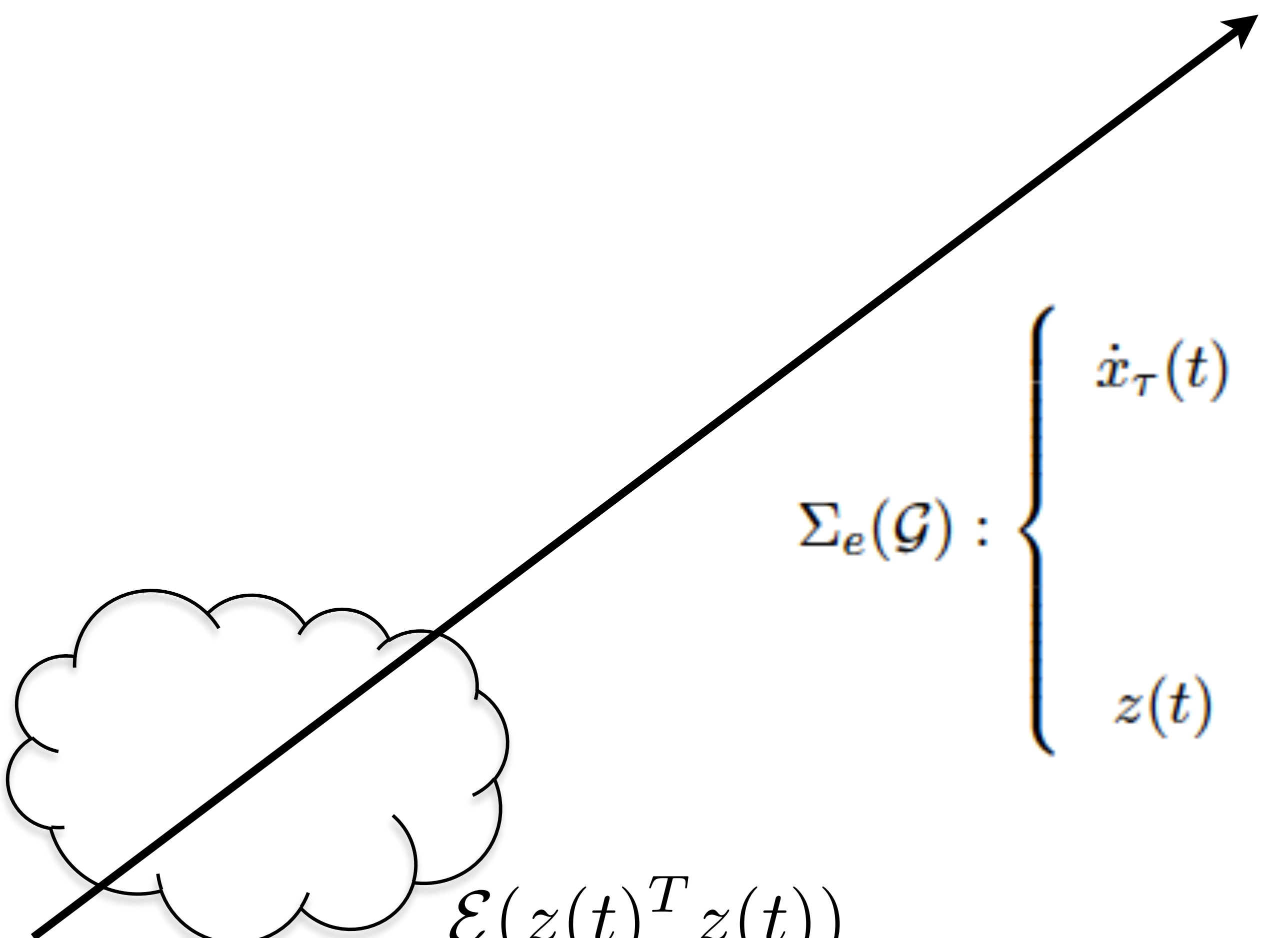


$$x_e(t) = \begin{bmatrix} E(\mathcal{T})^T \\ \frac{1}{n}\mathbf{1}^T \end{bmatrix} x(t)$$

$$\Sigma_e(\mathcal{G}) : \begin{cases} \dot{x}_\mathcal{T}(t) &= -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R^T_{(\mathcal{T},\mathcal{C})}x_\mathcal{T}(t) + \\ & \begin{bmatrix} E(\mathcal{T})^T & -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})} \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) &= x_\mathcal{T}(t). \end{cases}$$

stable and minimal realization of consensus protocol

Consensus with Exogenous Inputs



$$\Sigma_e(\mathcal{G}) : \begin{cases} \dot{x}_{\mathcal{T}}(t) &= -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R^T_{(\mathcal{T},\mathcal{C})}x_{\mathcal{T}}(t) + \\ & \begin{bmatrix} E(\mathcal{T})^T & -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})} \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) &= x_{\mathcal{T}}(t). \end{cases}$$

Performance of edge agreement problem can be used to study how noises affect the relative-state output

$\mathcal{H}_2$ Performance of Edge Agreement

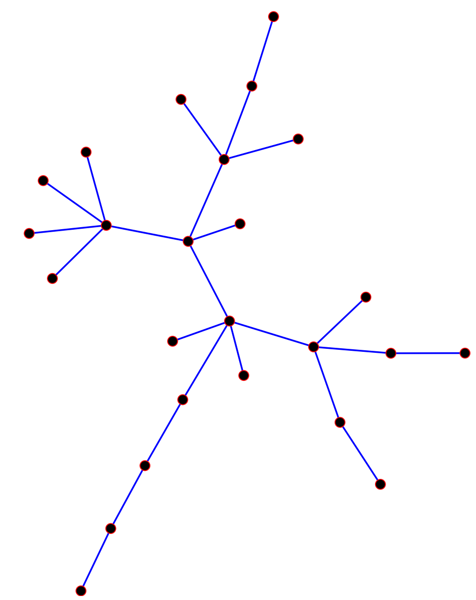
Theorem

$$\|\Sigma_e(\mathcal{G})\|_2^2 = \frac{1}{2}\mathbf{tr}\left[(R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T)^{-1}\right] + (n-1)$$

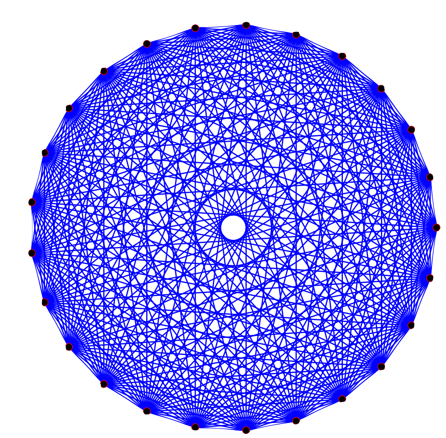
some immediate bounds...

$$\|\Sigma_e(\mathcal{G})\|_2^2 \leq \|\Sigma_e(\mathcal{T})\|_2^2 = \frac{3}{2}(n-1)$$

all trees are the same

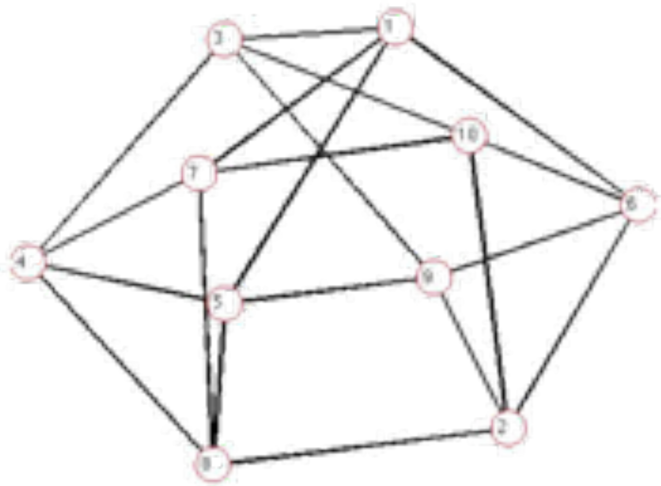


$$\|\Sigma_e(\mathcal{G})\|_2^2 \geq \|\Sigma_e(K_n)\|_2^2 = \frac{n^2-1}{n}$$

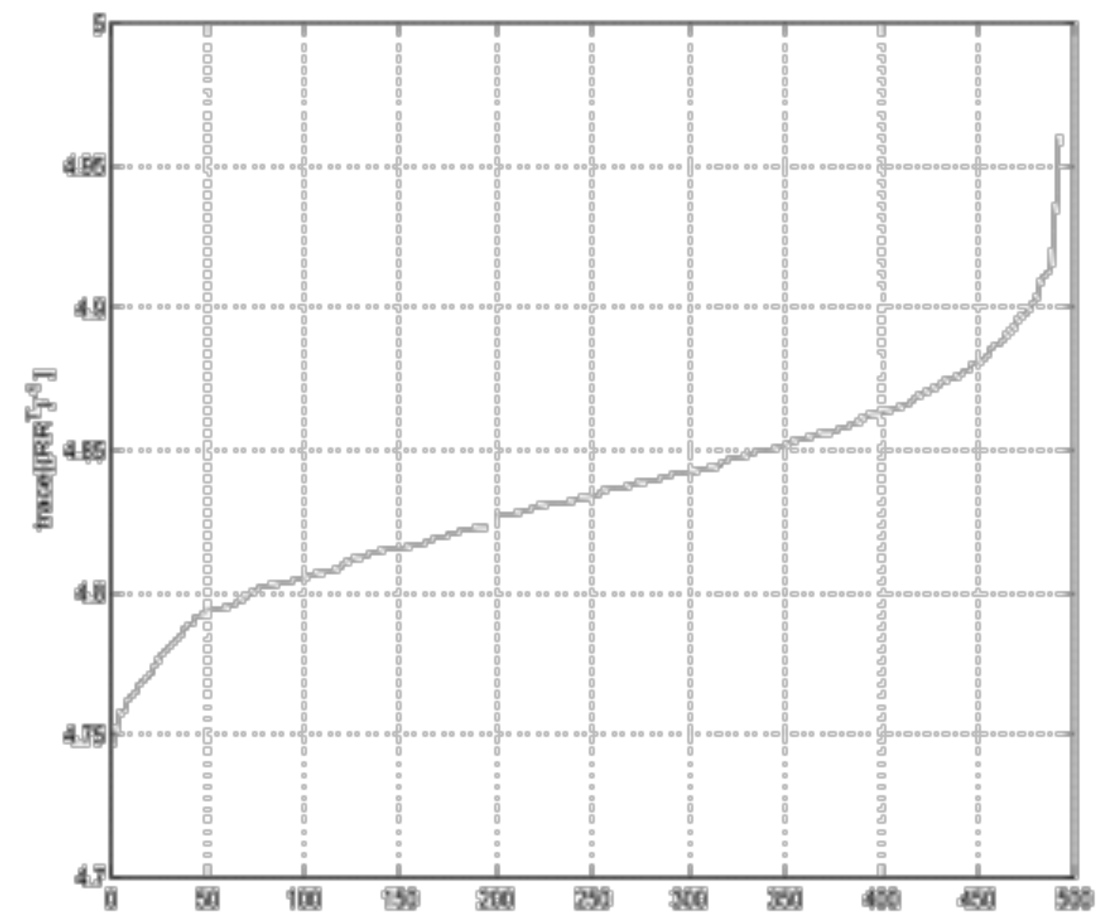


for general *k*-regular graphs…

$$\text{trace}\left[(RR^T)^{-1}\right]$$



dependant on structure
and cycles

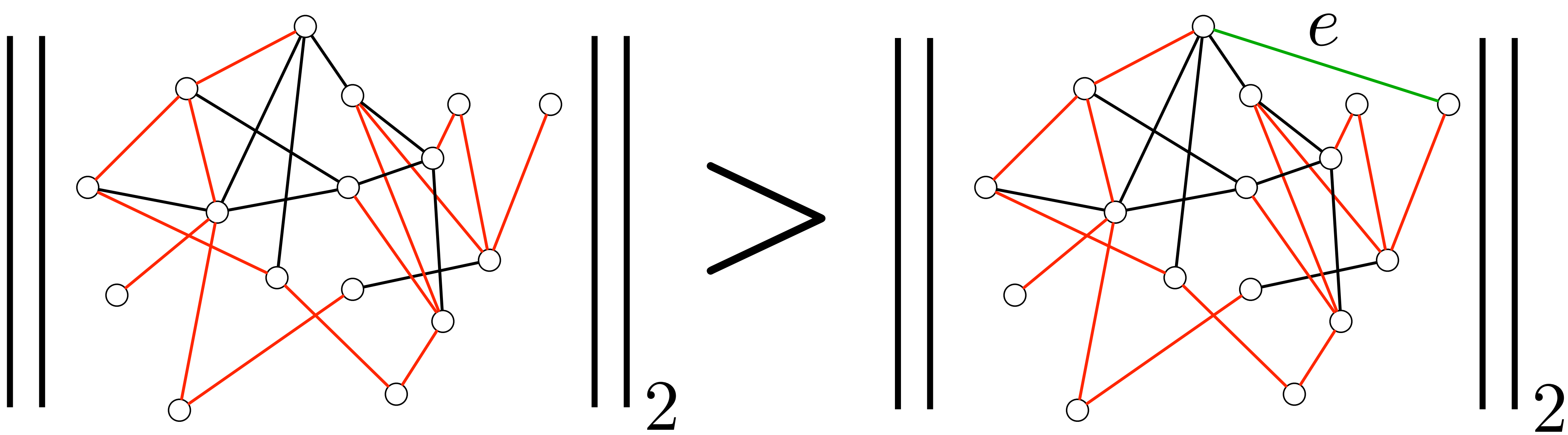


500 random 5-regular graphs

$\mathcal{H}_2$ Performance of Edge Agreement

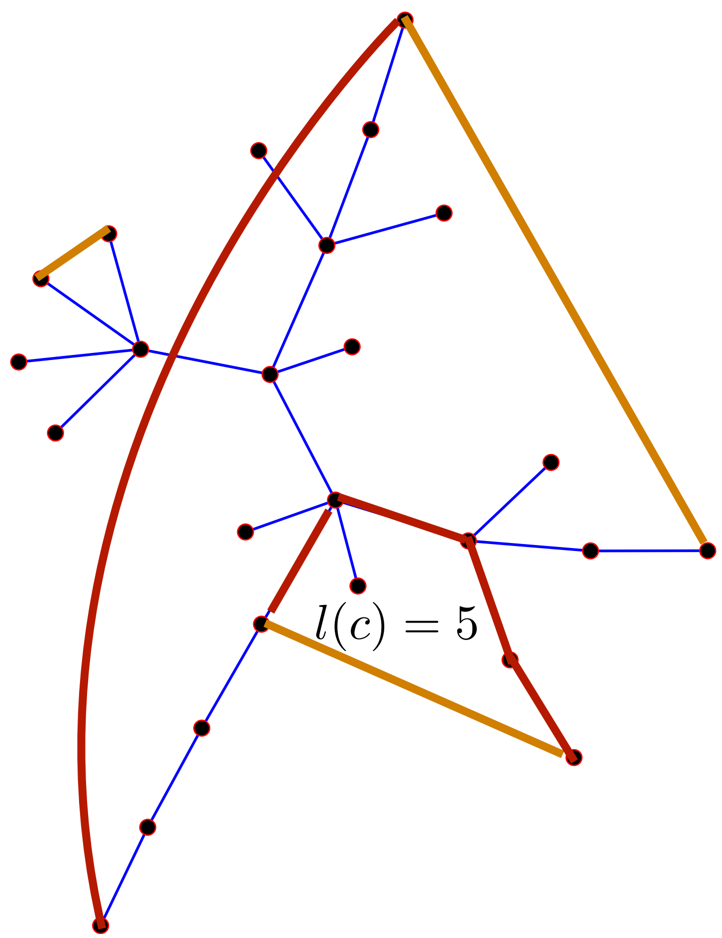
Theorem: Adding cycles always improves the performance.

$$\|\Sigma_e(\mathcal{G}\cup e)\|_2^2 = \|\Sigma_e(\mathcal{G})\|_2^2 - \frac{\left(R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} cc^T \left(R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1}}{2\left(1 + c^T \left(R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T\right)^{-1} c\right)}$$



Performance and Cycles

Is there a *combinatorial* feature
that affects the performance?

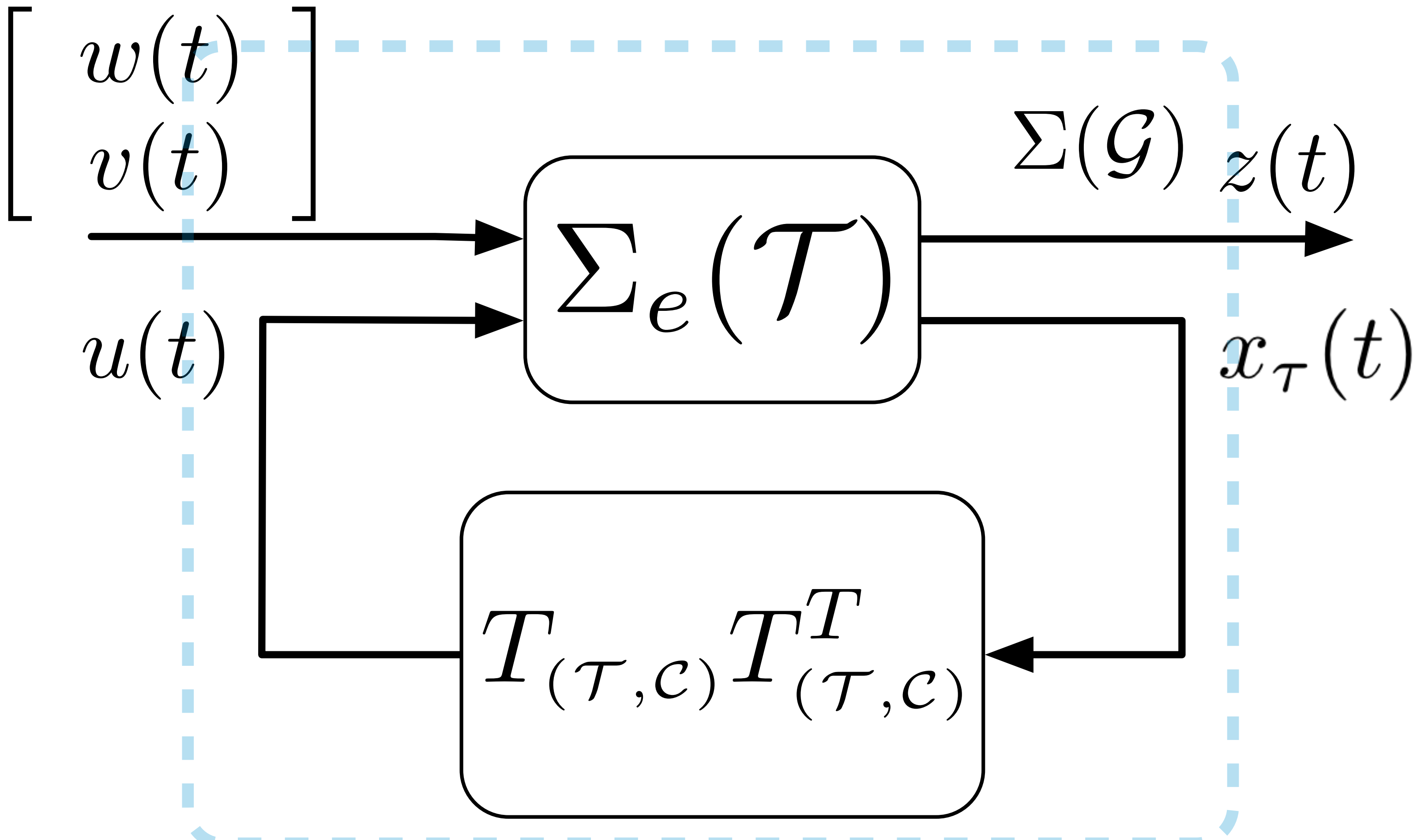


Corollary

$$\|\Sigma_e(\mathcal{T} \cup e)\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \frac{1}{2}(1 - l(c)^{-1})$$

long cycles are “better”

Cycles as Feedback

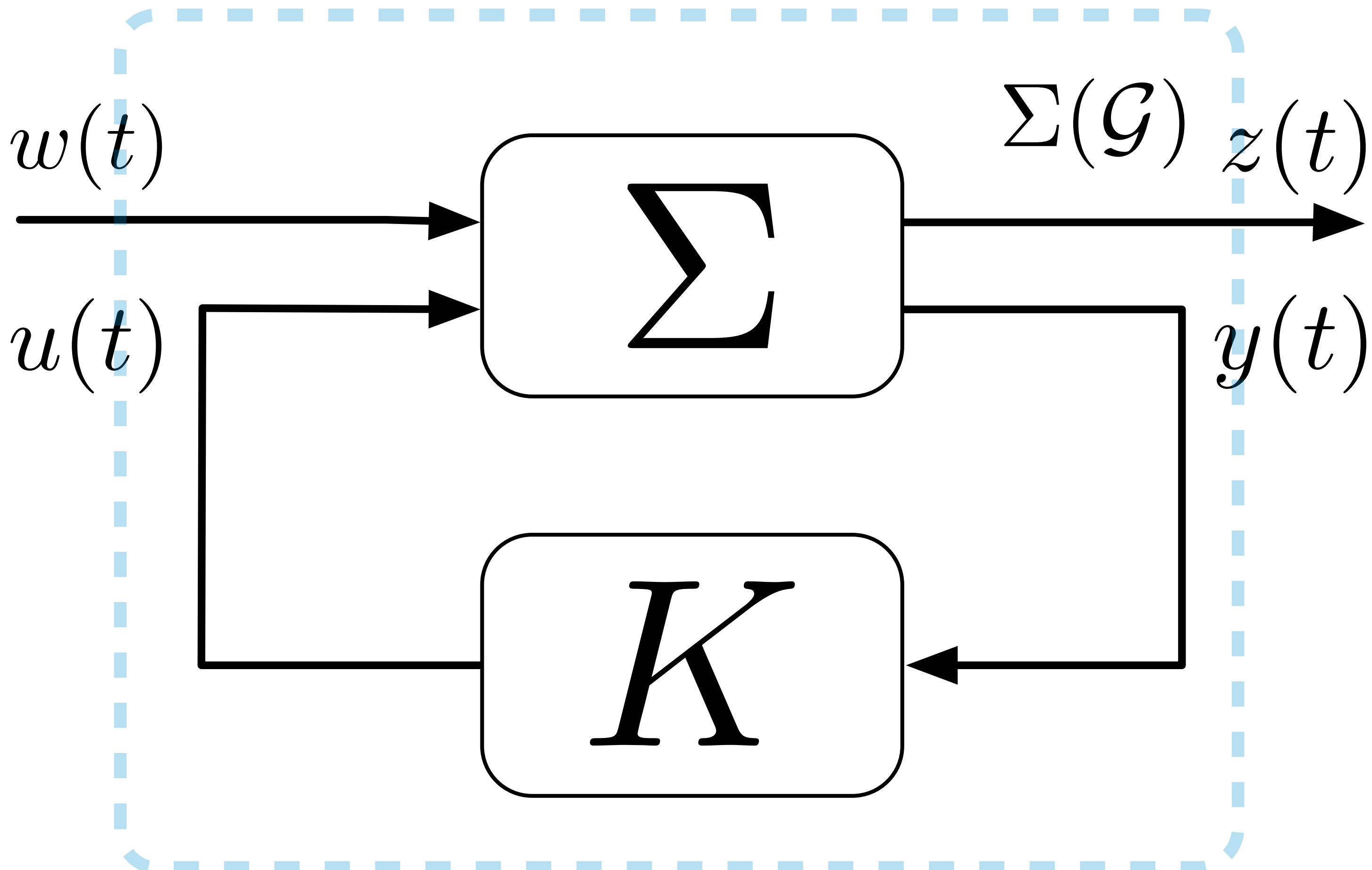


$$R_{(\mathcal{T},\mathcal{C})} = \begin{bmatrix} I & T_{(\mathcal{T},\mathcal{C})} \end{bmatrix}$$

$$E(\mathcal{T})T_{(\mathcal{T},\mathcal{C})} = E(\mathcal{C})$$

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R^T_{(\mathcal{T},\mathcal{C})} = L_e(\mathcal{T}) + L_e(\mathcal{T})\underline{T_{(\mathcal{T},\mathcal{C})}T^T_{(\mathcal{T},\mathcal{C})}}$$

Cycles as Feedback



$$R_{(\mathcal{T},\mathcal{C})} = \left[\begin{array}{cc} I & T_{(\mathcal{T},\mathcal{C})} \end{array} \right]$$

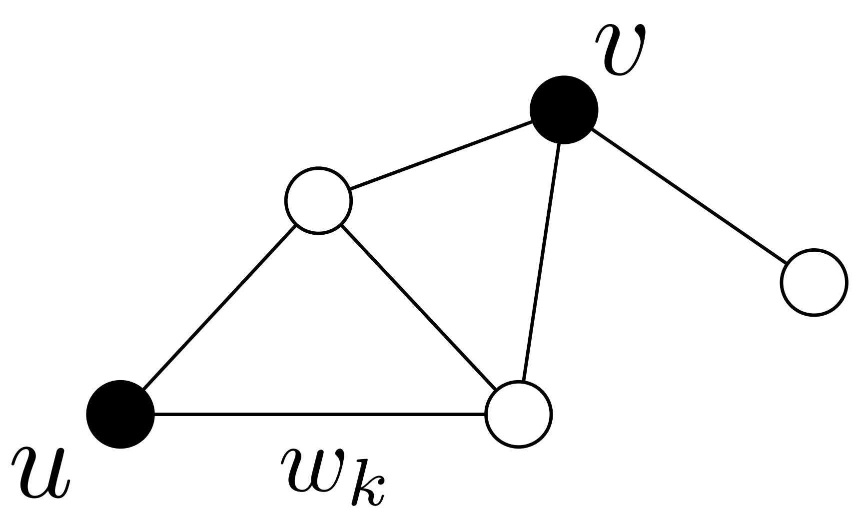
$$E(\mathcal{T})T_{(\mathcal{T},\mathcal{C})} = E(\mathcal{C})$$

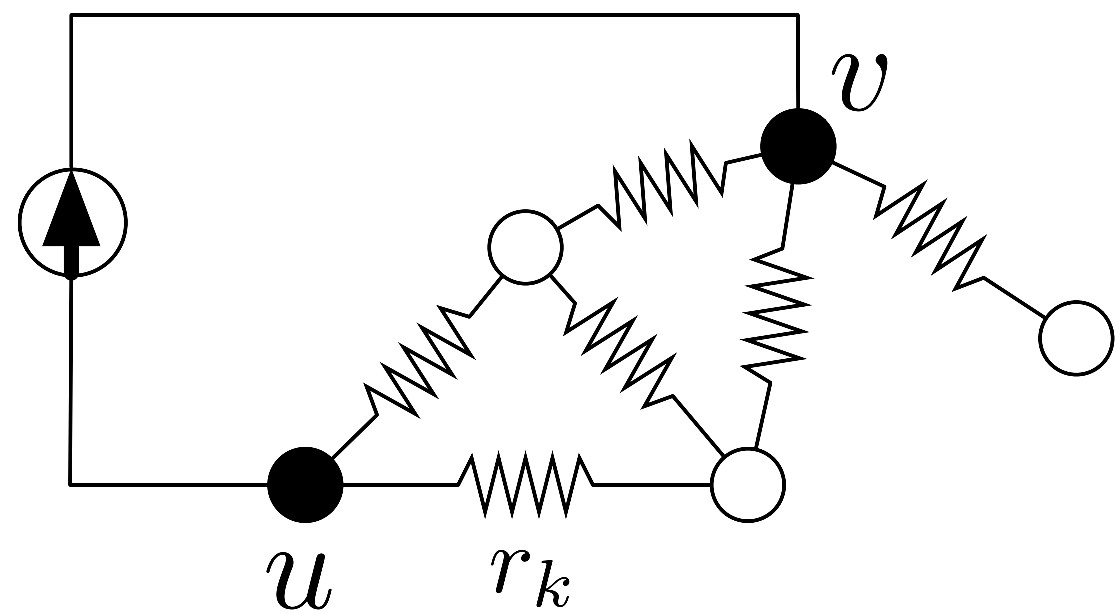
Design of consensus networks can be viewed as a *state-feedback problem*

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T = L_e(\mathcal{T}) + L_e(\mathcal{T})\underline{T_{(\mathcal{T},\mathcal{C})}T_{(\mathcal{T},\mathcal{C})}^T}$$

Effective Resistance of a Graph

The **effective resistance** between two nodes *u* and *v* is the electrical resistance measured across the nodes when the graph represents an electrical circuit with each edge a resistor





$$r_k = \frac{1}{w_k}$$ edge weights are the *conductance* of *each resistor*

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^\dagger(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$
$$= \left[L^\dagger(\mathcal{G})\right]_{uu} - 2\left[L^\dagger(\mathcal{G})\right]_{uv} + \left[L^\dagger(\mathcal{G})\right]_{vv}$$

Klein and Randić 1993

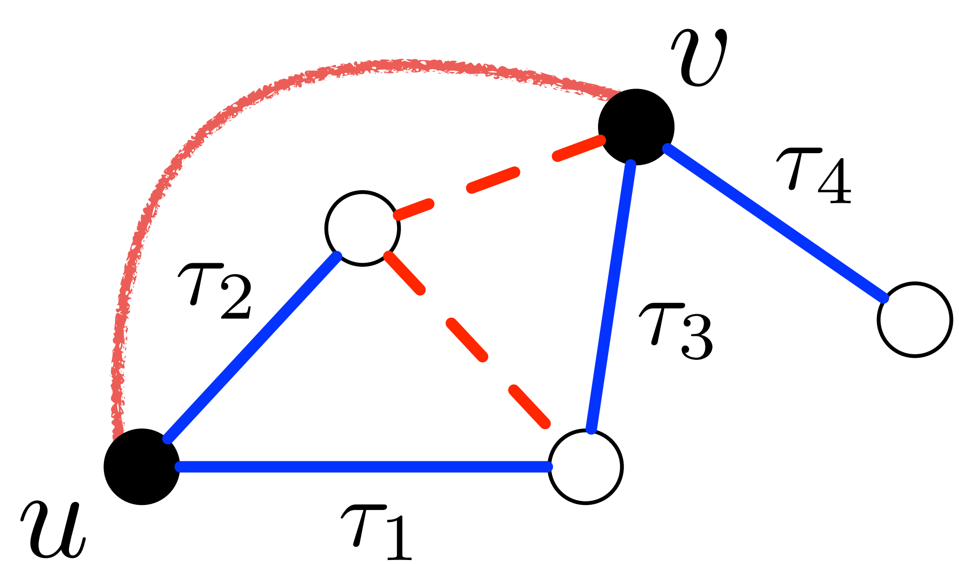
Effective Resistance of a Graph

Proposition 1

$$L^{\dagger}(\mathcal{G}) = (E_{\mathcal{T}}^{L})^{T} \left(R_{(\mathcal{T},\mathcal{C})} W R_{(\mathcal{T},\mathcal{C})}^{T} \right)^{-1} E_{\mathcal{T}}^{L}$$

$$r_{uv} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{\dagger}(\mathcal{G})(\mathbf{e}_u - \mathbf{e}_v)$$

$$E_{\mathcal{T}}^{L}(\mathbf{e}_u - \mathbf{e}_v) = \begin{bmatrix} \pm 1 \\ 0 \\ \pm 1 \\ 0 \end{bmatrix} \begin{matrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{matrix}$$



$$\mathcal{G} = \mathcal{T} \cup \mathcal{C}$$

indicates a path from node u to v using only edges in the spanning tree

$$T_{(\mathcal{T},\mathcal{C})} = \underbrace{(E_{\mathcal{T}}^{T} E_{\mathcal{T}})^{-1} E_{\mathcal{T}}^{T}}_{E_{\mathcal{T}}^{L}} E(\mathcal{C})$$

$\mathcal{H}_2$ Performance and Effective Resistance

Consensus driven by WGN

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & -L(\mathcal{G})x(t) + w(t) \\ z(t) & = & E(K_n)^T x(t) \end{array}\right.$$

monitor *all* possible relative states

$$\left[\begin{array}{c} x_{\mathcal{T}}(t) \\ \overline{x}(t) \end{array}\right] = \underbrace{\left[\begin{array}{c} E(\mathcal{T})^T \\ \frac{1}{n}\mathbb{1}^T \end{array}\right]}_{S^{-1}} x(t)$$

Edge Agreement….

$$\left\{\begin{array}{rcl} \dot{x}_{\mathcal{T}}(t) & = & -L_{ess}(\mathcal{G})x_{\mathcal{T}}(t) + E(\mathcal{T})^T w(t) \\ z(t) & = & E(K_n)^T E(\mathcal{T}) L_e(\mathcal{T})^{-1} x_{\mathcal{T}}(t) \end{array}\right.$$

$\mathcal{H}_2$ Performance and Effective Resistance

Edge Agreement....

$$\Sigma_e(\mathcal{G}) \left\{ \begin{array}{rcl} \dot{x}_\tau(t) & = & -L_{ess}(\mathcal{G})x_\tau(t) + E(\mathcal{T})^T w(t) \\ z(t) & = & E(K_n)^T E(\mathcal{T}) L_e(\mathcal{T})^{-1} x_\tau(t) \end{array} \right.$$

$$T_{(\mathcal{T},K_n)} = E(K_n)^T E(\mathcal{T}) L_e(\mathcal{T})^{-1}$$

$$\begin{aligned} \|\Sigma_e(\mathcal{G})\|_2^2 &= \frac{1}{2}\mathrm{tr}\left[T^T_{(\mathcal{T},K_n)} (\mathcal{R}_{(\mathcal{T},\mathcal{C})} \mathcal{R}^T_{(\mathcal{T},\mathcal{C})})^{-1} T_{(\mathcal{T},K_n)} \right] \\ &= \frac{1}{2} R_{tot}(\mathcal{G}) \end{aligned}$$