Analysis and Control of Multi-Agent Systems

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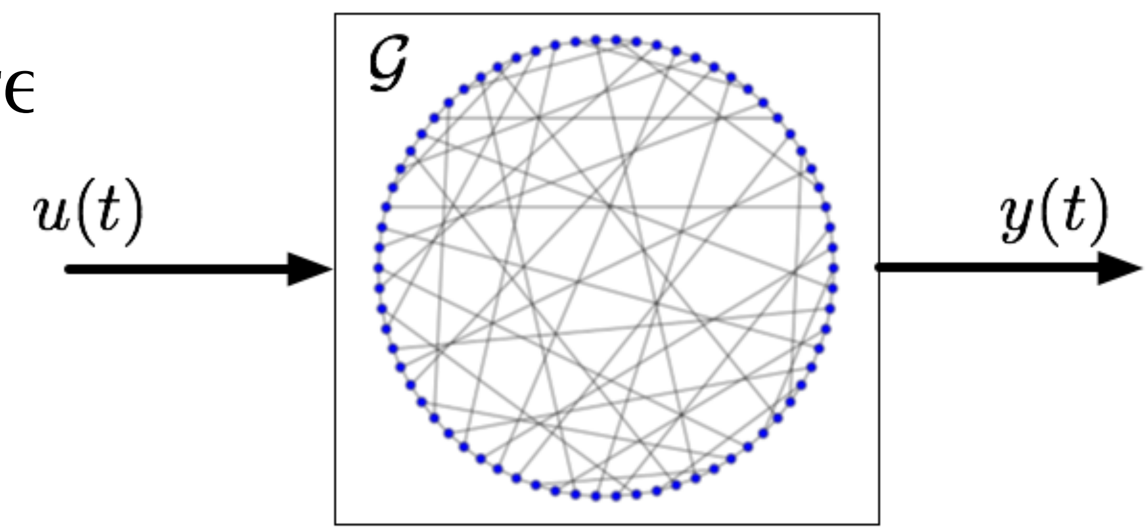
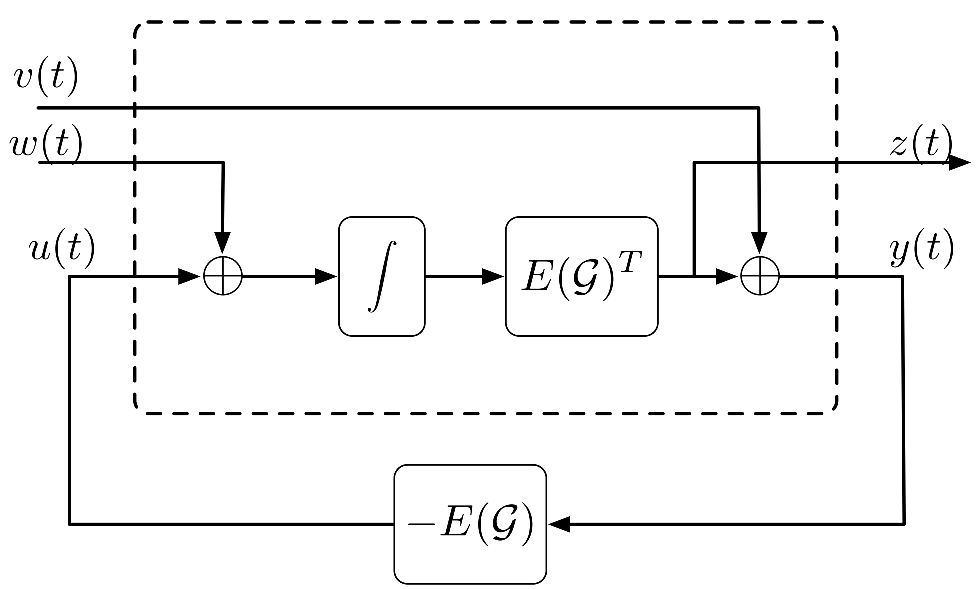
Network Synthesis

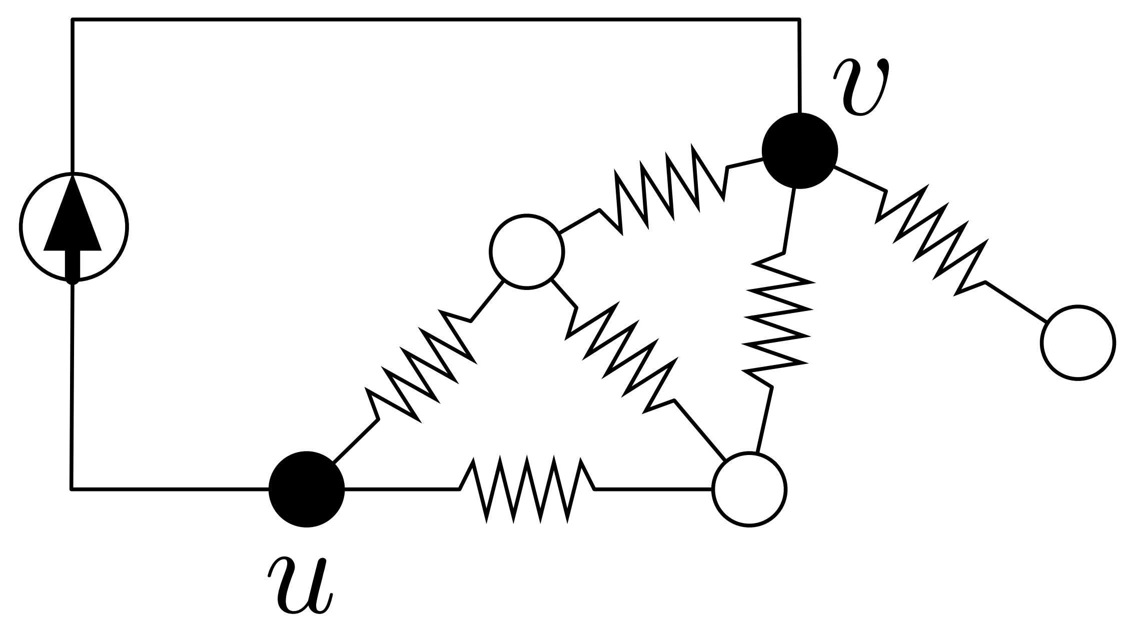
last time…

Performance of Consensus

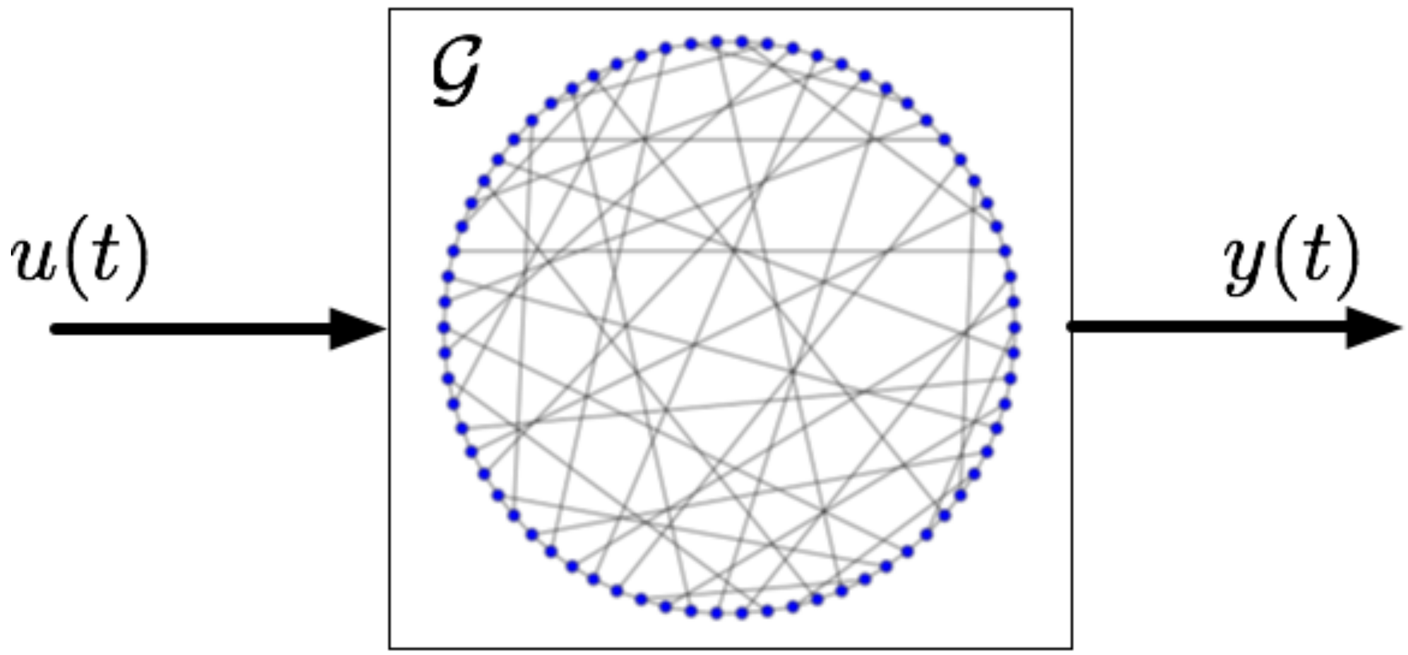
- H2 system performance
- performance depends on structure of graph (i.e., cycles)
- relation to effective resistance







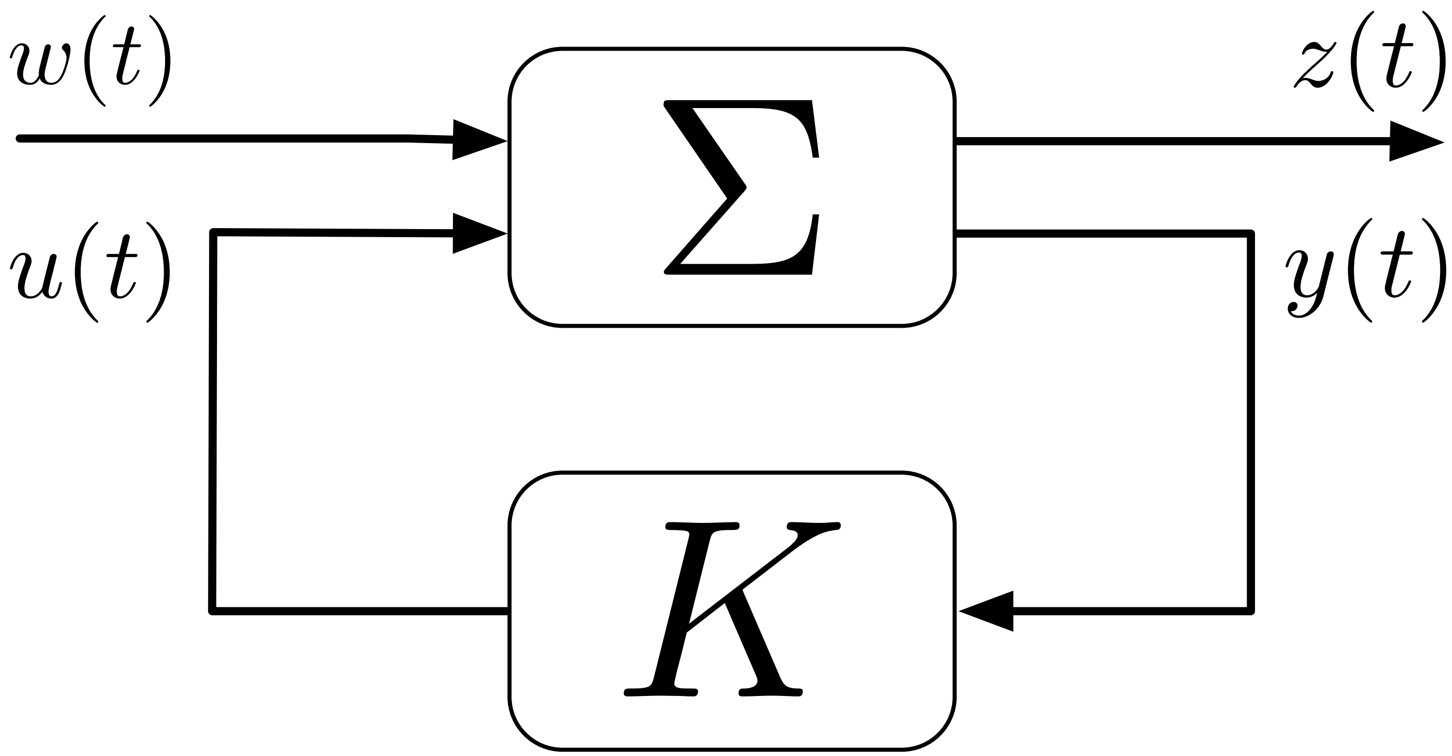
Design of Networks



graph-theoretic properties that influence system performance

- spanning trees, rooted out-branchings
- algebraic connectivity
- cycles
- graph symmetry

The Modern Control Paradigm



$$\min_{K} \;\|\Sigma_{cl}\|_p$$

$$s.t.\; K \text{ is stabilizing}$$

The Modern Control Paradigm

The (infinite-horizon) Linear Quadratic Regulator (LQR)

$$\begin{array}{cc} \min_{x,u} & \int_0^{\infty} (x^T Q x + u^T R u) dt \\ \text{s.t.} & \dot{x} = Ax + Bu \\ & x(0) = x_0 \end{array}$$

linear state-feedback is the *optimal* solution

$$u(t) = Kx(t)$$

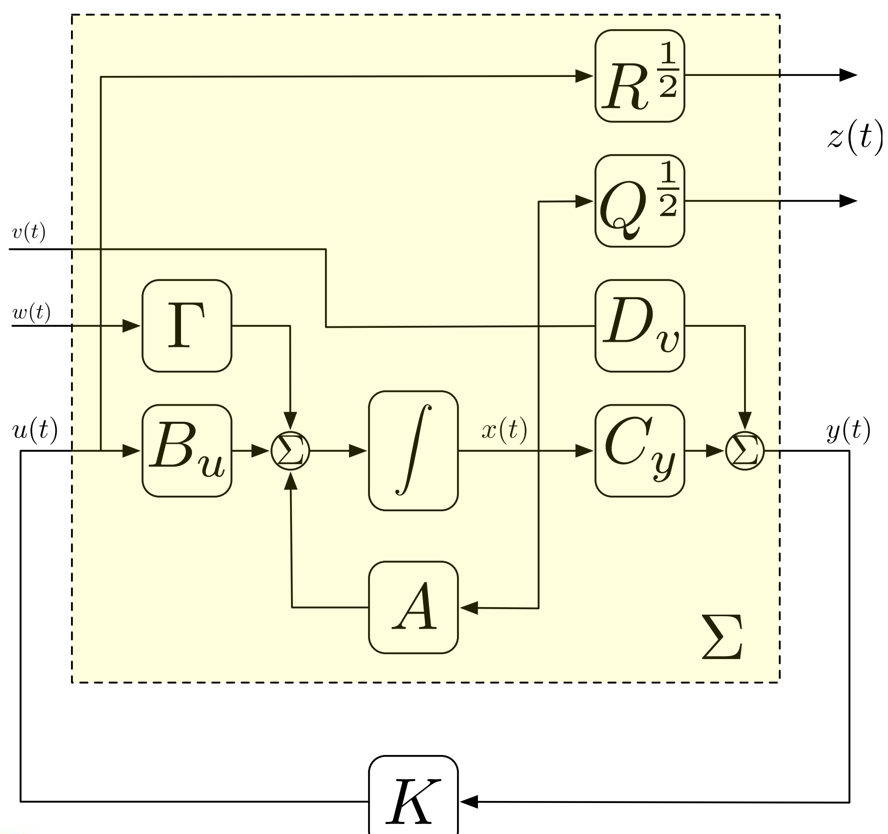
$$\dot{x}(t) = (A + BK)x(t)$$

- infinite dimensional

The Modern Control Paradigm

Optimal $\mathcal{H}_2$ Control

$$\begin{array}{ll} \min_{W,X,Z} & \mathrm{trace}[W] \\ s.t. & \begin{bmatrix} A & B_u \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix}\begin{bmatrix} A^T \\ B_u^T \end{bmatrix} + \Gamma\Gamma^T < 0 \\ & \begin{bmatrix} X & (Q^{\frac{1}{2}}X + R^{\frac{1}{2}}Z)^T \\ (Q^{\frac{1}{2}}X + R^{\frac{1}{2}}Z) & W \end{bmatrix} > 0 \end{array}$$



$$\min_K \ \|\Sigma_{cl}\|_p$$

$s.t.$ K is stabilizing

equivalent to LQR solution!

- finite dimensional

Static Optimization

A *static* optimization problem

$$\min_x f(x)$$

$$\text{s.t.} \begin{cases} g_i(x) \leq 0, & i = 1, \ldots, m \\ h_j(x) = 0, & j = 1, \ldots, r \\ x \in \mathcal{X} & \end{cases}$$

convex v. non-convex

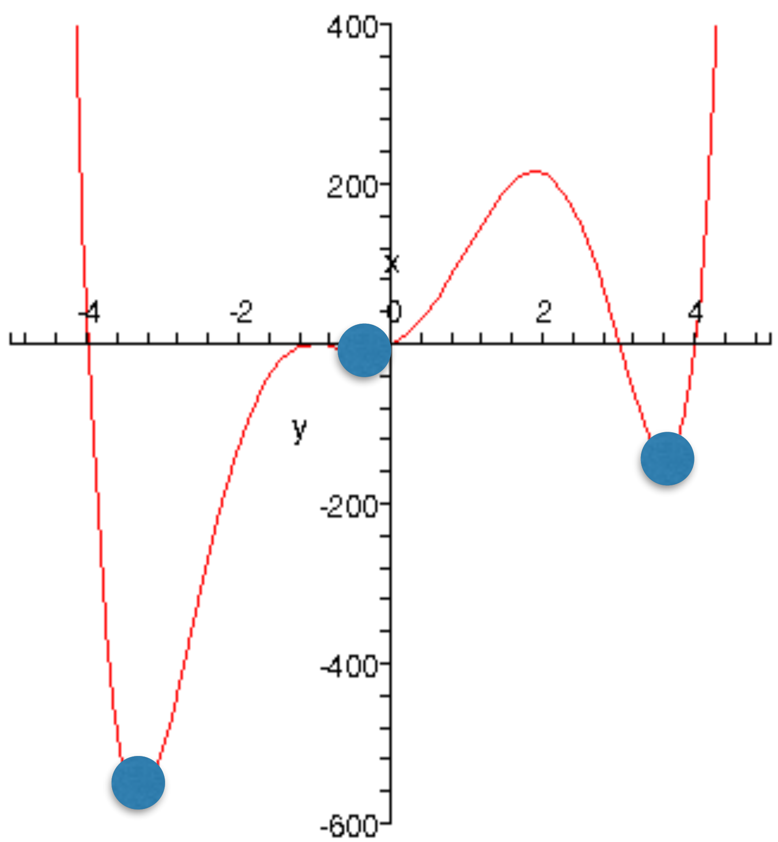
Prof. R. T. Rockafellar

"…in fact, the great watershed in optimization isn't between linearity and non-linearity, but convexity and non-convexity"

Convex Optimization

when objective function is *convex* $\min_x f(x)$

- any local minima is a global minimum



non-convex - "local" minima

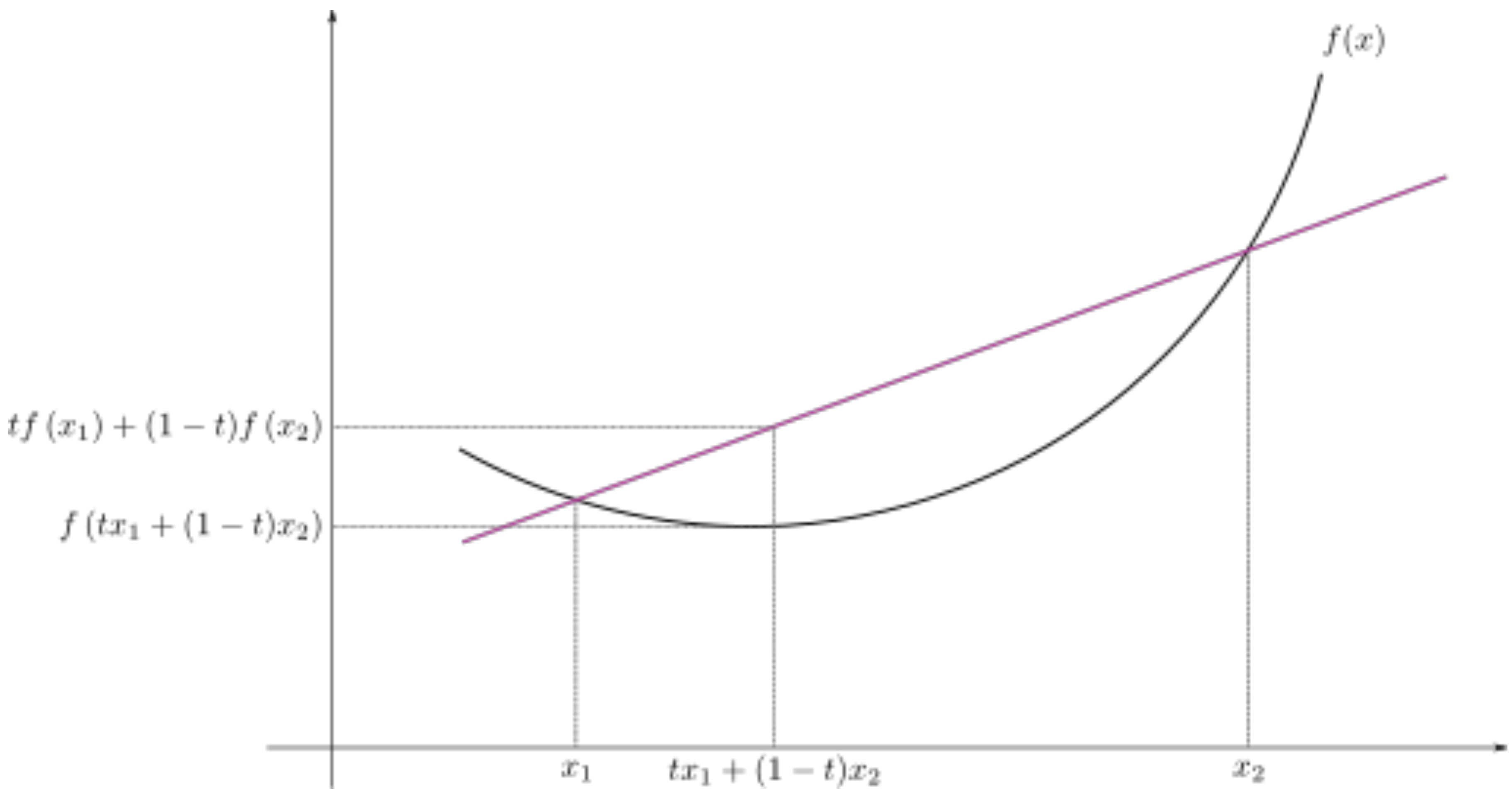
convex - "local" minima is global

Convex Optimization

Definition

a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if for any two points $x_1, x_2 \in \mathbb{R}^n$ and for any $\lambda \in [0, 1]$, the following inequality holds:

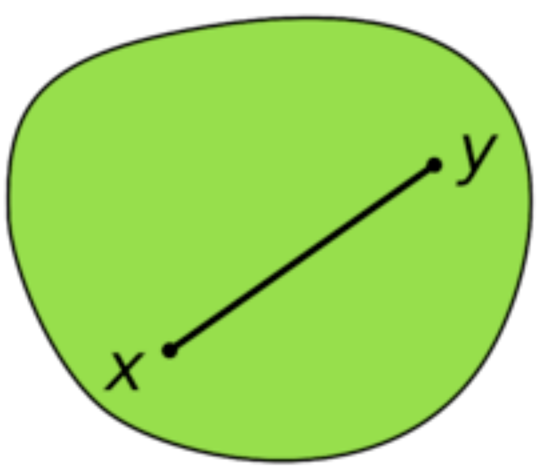
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$



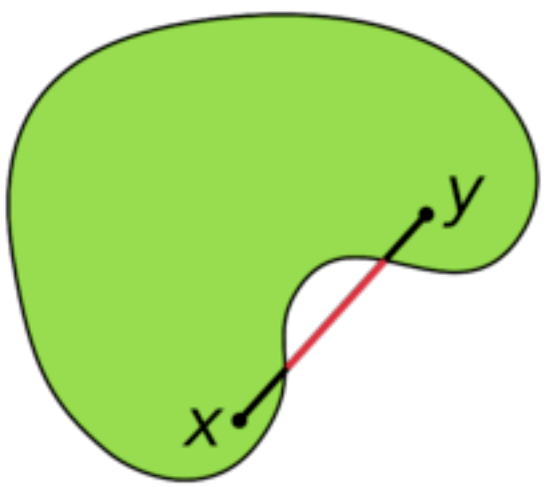
Convex Optimization

Definition

A set $S \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in S$ and $\lambda \in [0, 1]$, the point $(1-\lambda)x + \lambda y \in S$.



convex



non-convex

Convex Optimization

A *static* optimization problem

$$\min_x f(x)$$

$$\text{s.t.} \begin{cases} g_i(x) \leq 0, & i = 1, \ldots, m \\ h_j(x) = 0, & j = 1, \ldots, r \\ x \in \mathcal{X} \end{cases}$$

convex if:

$f(x)$

$g_i(x),\ i = 1, \ldots, m$

$h_j(x),\ j = 1, \ldots, r$

$\mathcal{X}$

are all convex

Convex Optimization

semi-definite programming (SDP)

$$\begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & F(x) = F_0 + \sum_{i=1}^m x_i F_i \geq 0 \end{array}$$

F_i - symmetric matrices

$F(x) \geq 0 \Leftrightarrow z^T F(x) z \geq 0, \forall z$ - Linear Matrix Inequality (LMI)

$A \geq B \Leftrightarrow A - B \geq 0$ - positive (semi) definite ordering

$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \geq 0,\ C > 0 \Leftrightarrow A - BC^{-1}B^T \geq 0$ - Schur Complement

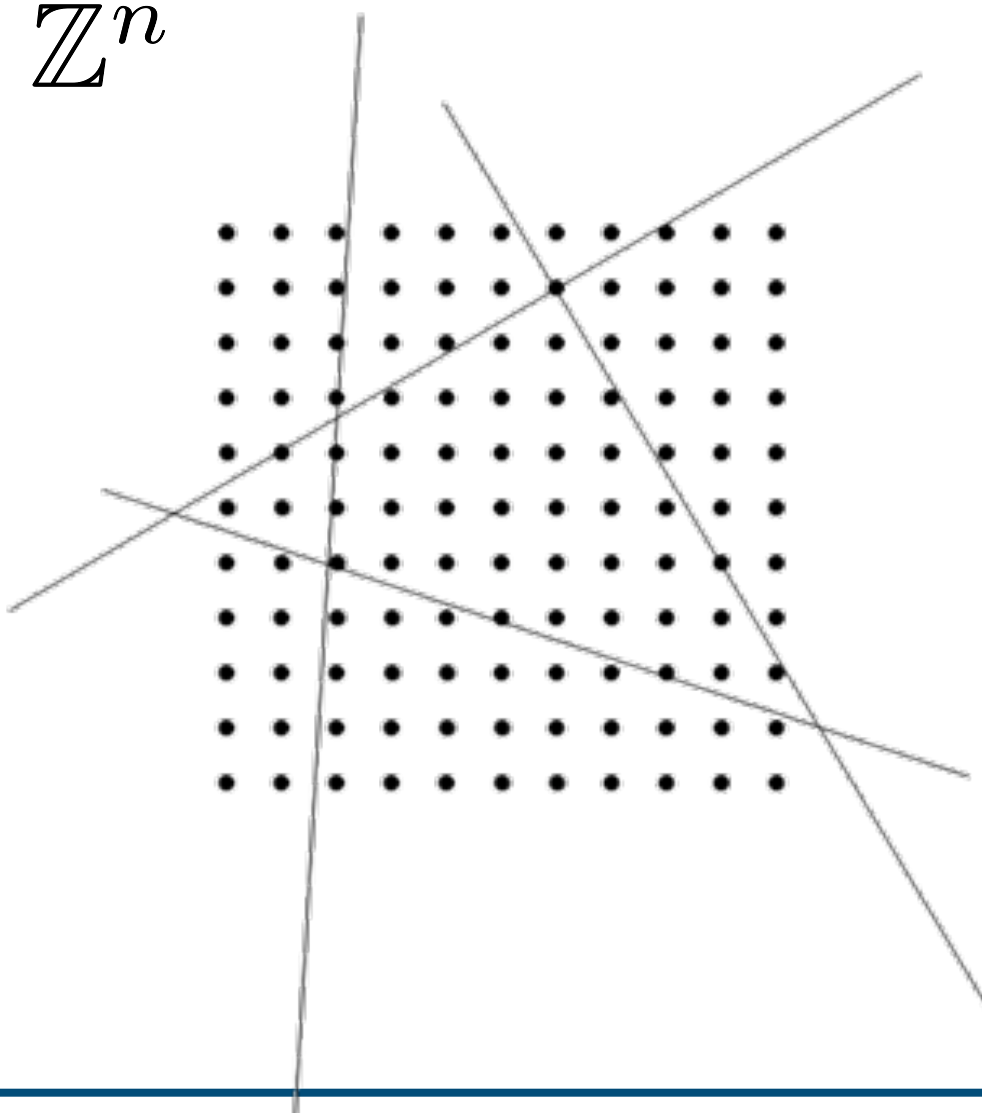
Non-Convex Optimization

integer programming

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

optimization variable is constrained to be an integer

example: $x \in \{0, 1\}$

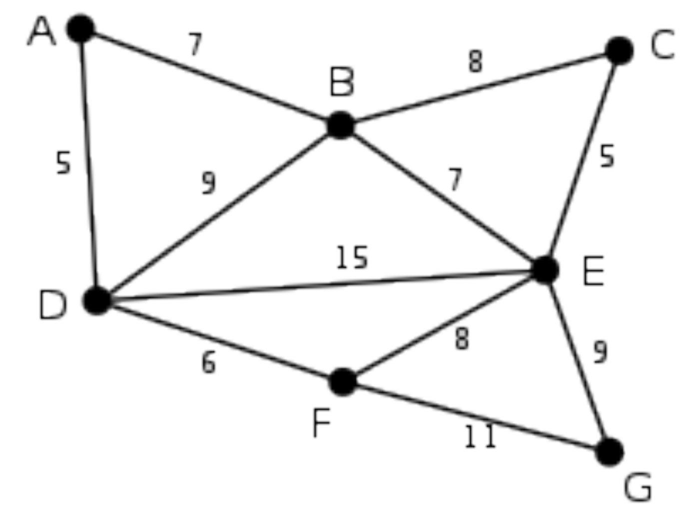


Minimum Weight Spanning Tree

problem data:

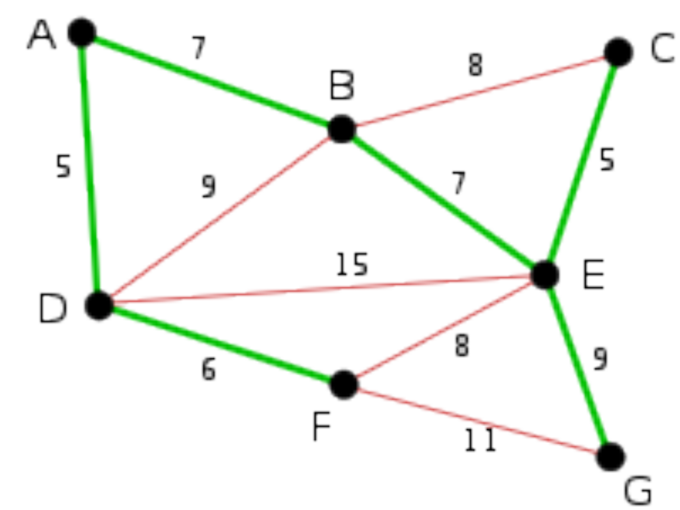
- a connected and undirected graph
- positive weights on each edge

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$



goal: Find a spanning tree of minimum weight

$$\min_{T} \sum_{e \in T} w_e$$
$$\text{s.t.}\, T \in \mathcal{T}$$

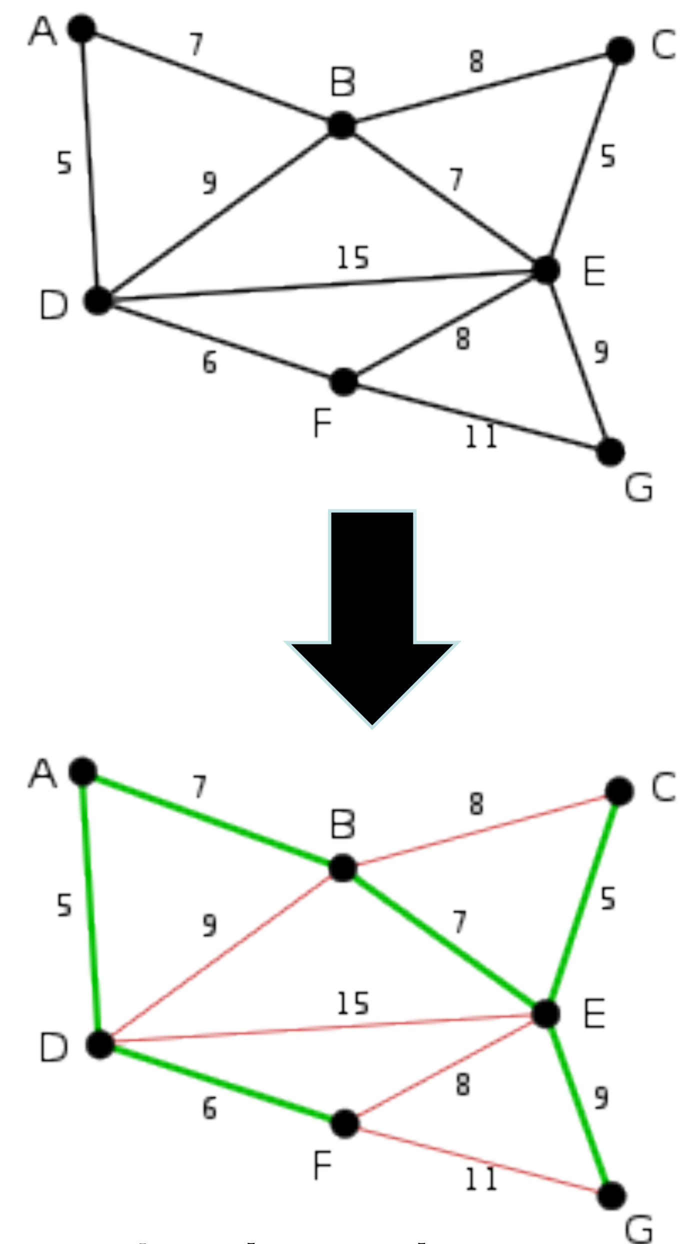


$\mathcal{T}$ - set of all spanning trees

Minimum Weight Spanning Tree

$$\min_T \sum_{e \in T} w_e$$
$$\text{s.t.} T \in \mathcal{T}$$

solution: Kruskal's Algorithm



Algorithm 1: Kruskal's Algorithm

Data: A connected undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and weights $w : \mathcal{E} \mapsto \mathbb{R}$.

Result: A spanning tree $\mathcal{G}_t$ of minimum weight.

begin

Sort the edges such that

$w(e_1) \leq w(e_2) \leq \cdots \leq w(e_{|\mathcal{E}|})$, where $e_i \in \mathcal{E}$

Set $\mathcal{G}_t := \mathcal{G}_t(\mathcal{V}, \emptyset)$

for $i := 1$ **to** $|\mathcal{E}|$ **do**

if $\mathcal{G}_t + e_i$ *contains no cycle* **then**

set $\mathcal{G}_t := \mathcal{G}_t + e_i$

---

a *greedy* algorithm

$\mathcal{O}(|\mathcal{E}| \log |\mathcal{V}|)$

Design of RSN Sensing Graphs

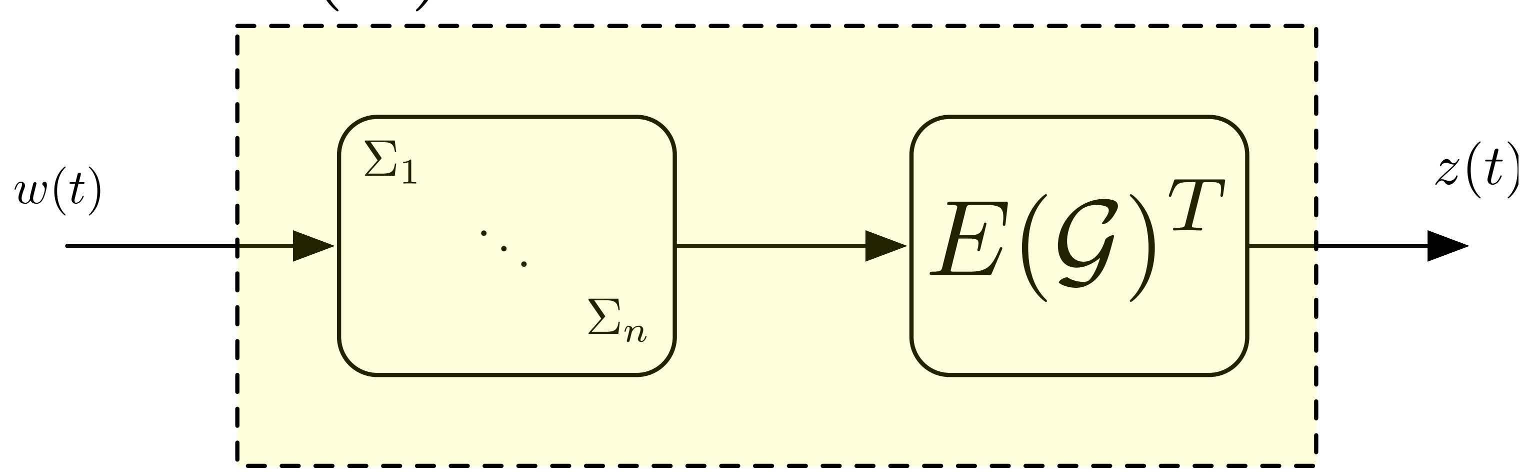
Relative Sensing Network with *heterogeneous* agent dynamics

$$\Sigma_i : \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i w(t) \\ y_i(t) &= C_i x_i(t) \end{cases} \qquad i = 1, \ldots, n$$

relative outputs (on edges)

$$z(t) = (E(\mathcal{G})^T \otimes I_q) \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_n \end{bmatrix} x(t)$$

$\Sigma(\mathcal{G})$



Matrix Kronecker Product

$$A \in \mathbb{R}^{n\times m}$$

$$B \in \mathbb{R}^{p\times q}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}$$

examples

$$\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} = I \otimes A$$

$$\mathbb{1}_n \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix Kronecker Product

$$A \in \mathbb{R}^{n\times m}$$
$$B \in \mathbb{R}^{p\times q}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}$$

properties

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

$$A = U_A \Sigma_A V_A^T$$
$$B = U_B \Sigma_B V_B^T$$

$$(A \otimes B) = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)^T$$

Design of RSN Sensing Graphs

Performance of each agent

$$\|\Sigma_i\|_2^2 = \mathrm{tr}[C_i P_i C_i^T]$$

$$A_i P_i + P_i A_i^T + B_i B_i^T = 0$$

Theorem

$$\|\Sigma(\mathcal{G})\|_2^2 = \sum_{i=1}^{n} d_i \|\Sigma_i\|_2^2$$

how to *design* a connected sensing graph with smallest $\mathcal{H}_2$ performance?

Design of RSN Sensing Graphs

a synthesis problem

$$\min_{\mathcal{G}} \ \|\Sigma(\mathcal{G})\|_2^2$$

$$s.t.\ \mathcal{G} \subset \mathbf{G},\ \mathcal{G} \text{ connected}$$

a *combinatorial* optimization problem!

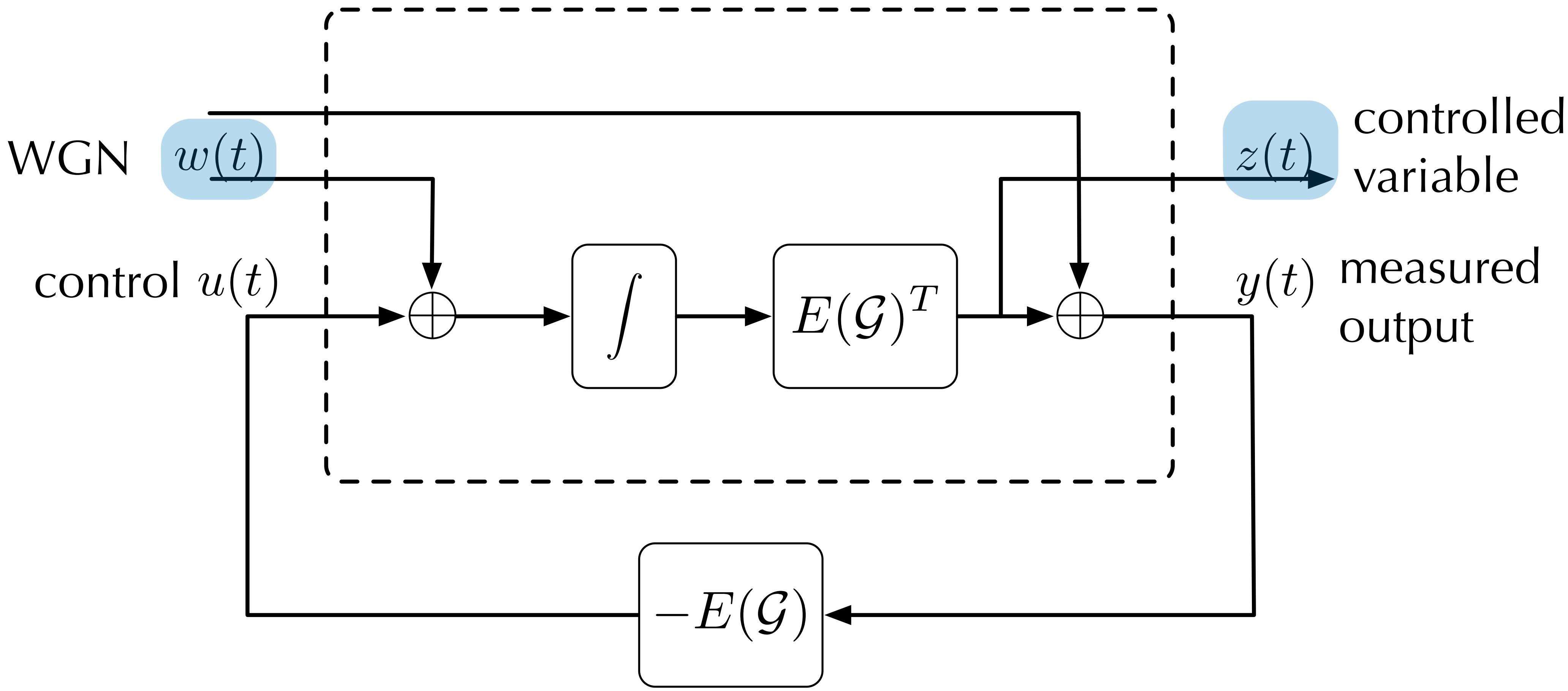
Theorem

The RSN synthesis problem is equivalent
to the minimum weight spanning tree problem.

proof

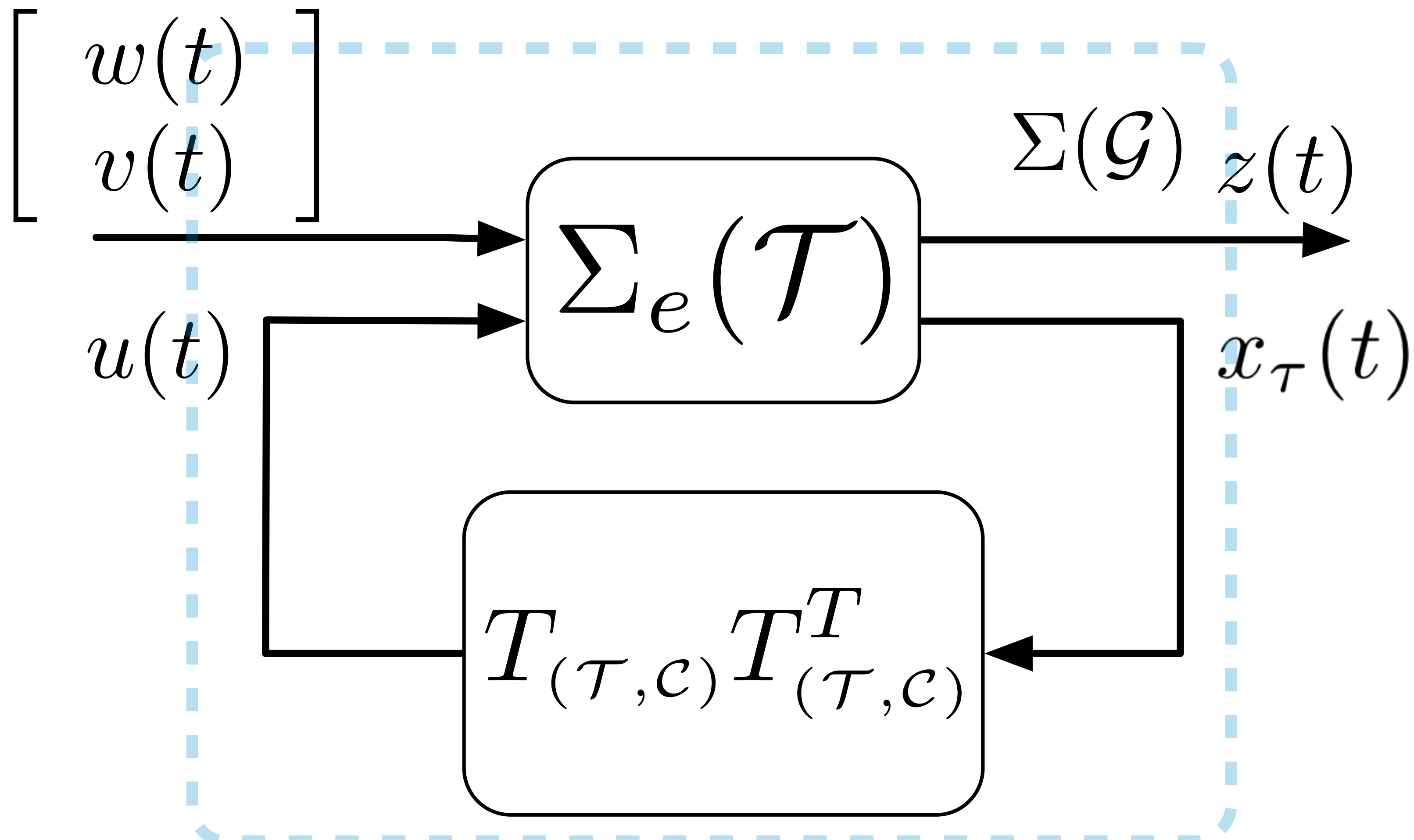
convert objective function to an equivalent
function on the weights of an associated weighted graph

Design of Cycles in Consensus



Given a nominal graph, we would like to add a fixed number of edges that lead to the largest improvement in the $\mathcal{H}_2$ performance of the system.

Cycles as Feedback



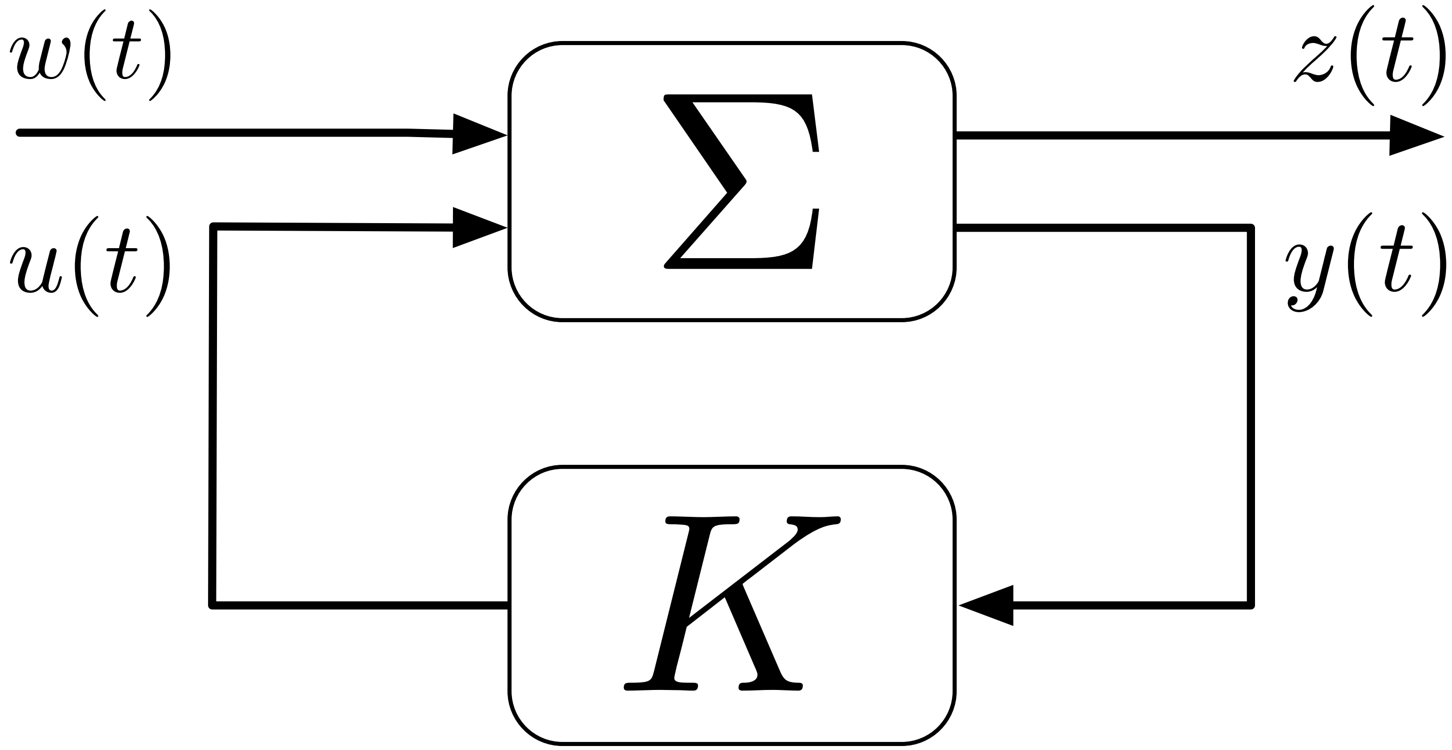
$$R_{(\mathcal{T},\mathcal{C})} = \begin{bmatrix} I & T_{(\mathcal{T},\mathcal{C})} \end{bmatrix}$$

$$E(\mathcal{T})T_{(\mathcal{T},\mathcal{C})} = E(\mathcal{C})$$

Design of consensus networks can be viewed as a *state-feedback problem*

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T = L_e(\mathcal{T}) + L_e(\mathcal{T})\underline{T_{(\mathcal{T},\mathcal{C})}T_{(\mathcal{T},\mathcal{C})}^T}$$

Cycles as Feedback



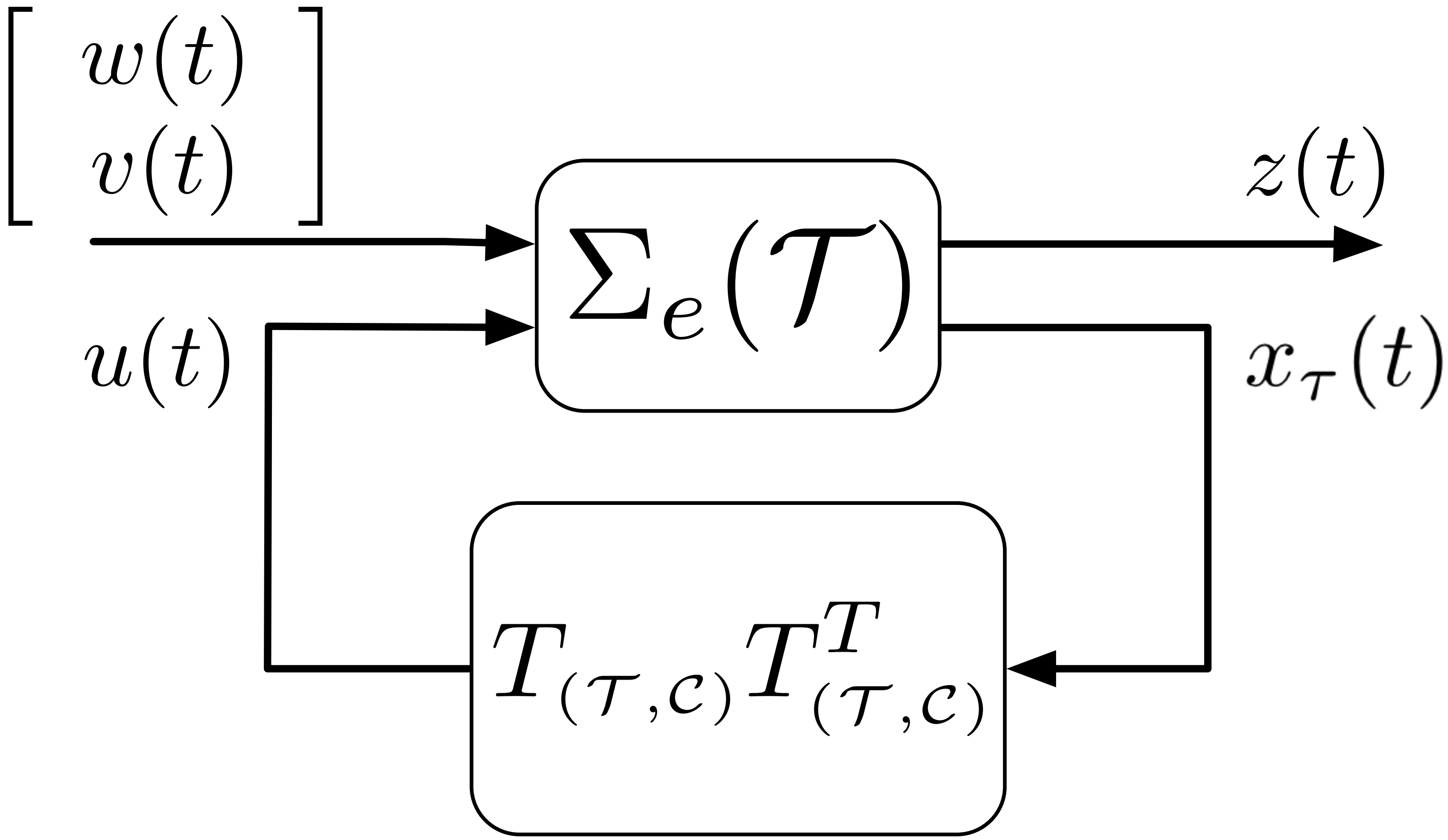
$$R_{(\mathcal{T},\mathcal{C})} = \left[\begin{array}{cc} I & T_{(\mathcal{T},\mathcal{C})} \end{array} \right]$$

$$E(\mathcal{T})T_{(\mathcal{T},\mathcal{C})} = E(\mathcal{C})$$

Design of consensus networks can be viewed as a *state-feedback problem*

$$L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R^T_{(\mathcal{T},\mathcal{C})} = L_e(\mathcal{T}) + L_e(\mathcal{T})\underline{T_{(\mathcal{T},\mathcal{C})}T^T_{(\mathcal{T},\mathcal{C})}}$$

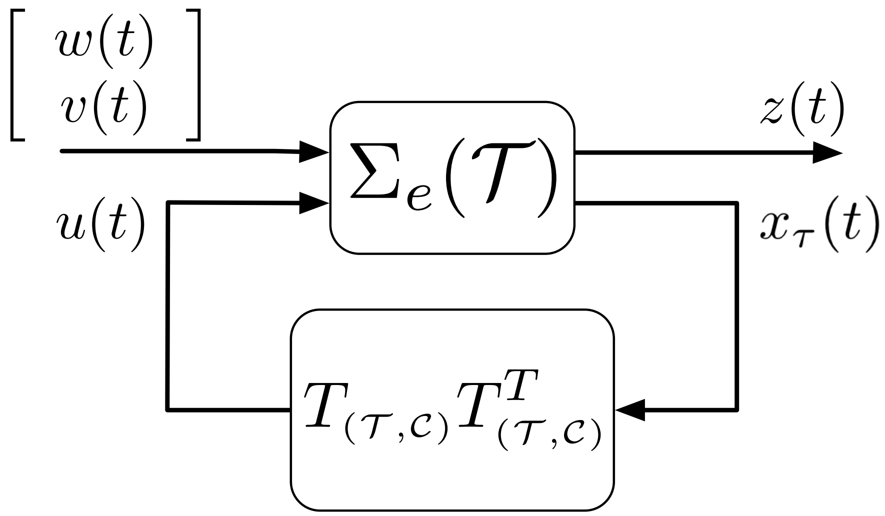
Design of Cycles in Consensus



A synthesis problem

$$\min_{T_{(\mathcal{T},\mathcal{C})}\in\mathbb{R}^{|\mathcal{V}|\times k}} \|\Sigma_e(\mathcal{G})\|_2,$$

Design of Cycles



$$\min_{T_{(\mathcal{T},\mathcal{C})}\in\mathbb{R}^{|\mathcal{V}|\times k}} \|\Sigma_e(\mathcal{G})\|_2,$$

Given a spanning tree, add **k** edges that maximize the performance improvement

a mixed-integer SDP

$$\begin{aligned} \min_{M,w_i} \quad & \textbf{trace}\,[M] \\ \text{s.t.} \quad & \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0 \\ & \sum_i w_i = k,\ w_i \in \{0,1\} \end{aligned}$$

Design of Cycles

a mixed-integer SDP

$$\min_{M,w_i} \quad \textbf{trace}\,[M]$$

$$\text{s.t.} \quad \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0$$

$$\sum_i w_i = k, \; \cancel{w_i \in \{0,1\}} \qquad w_i \in [0,1]$$

relaxation to *weighted* edges "misses the point"

$$\min_{M,w_i} \quad \textbf{trace}[M] + \text{card}(w)$$

$$\text{s.t.} \quad \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0$$

$$\sum_i w_i = k, \; w_i \in [0,1]$$

attempt to minimize "# of non-zero elements"

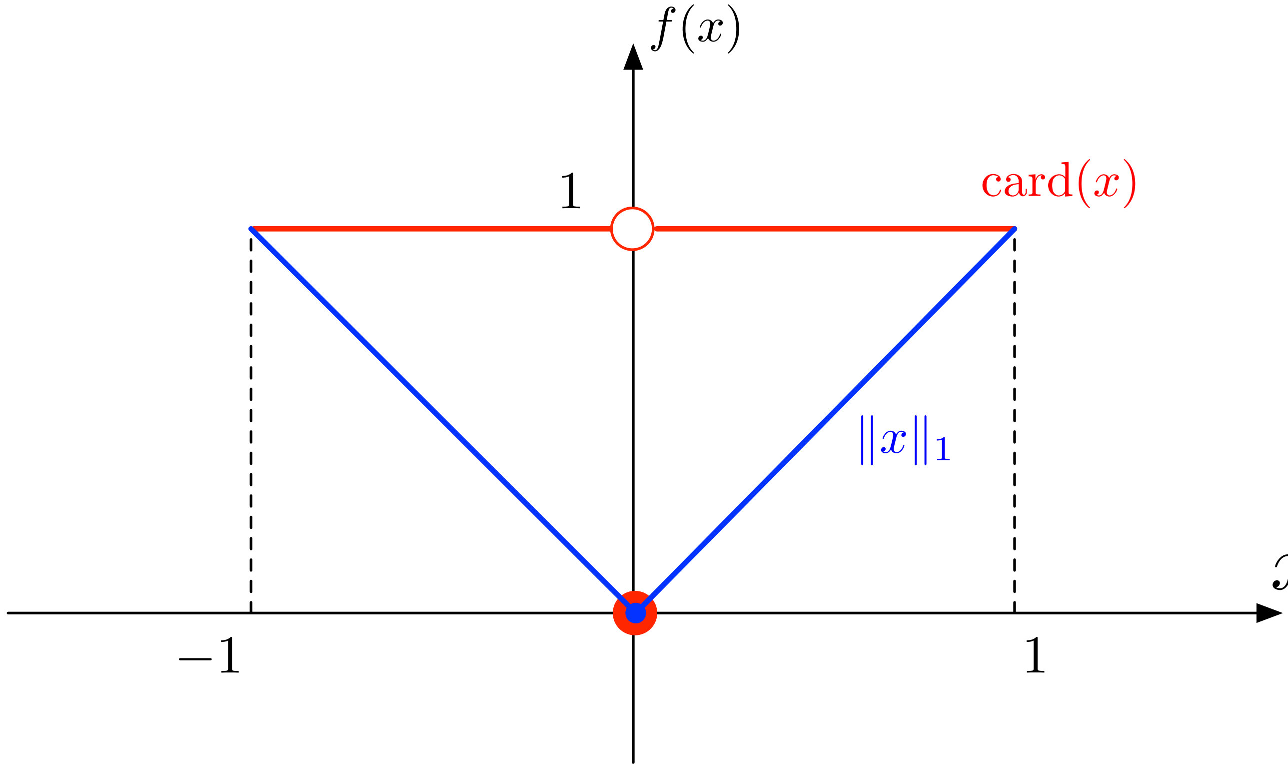
not a convex relaxation!

Convex Envelope of Cardinality

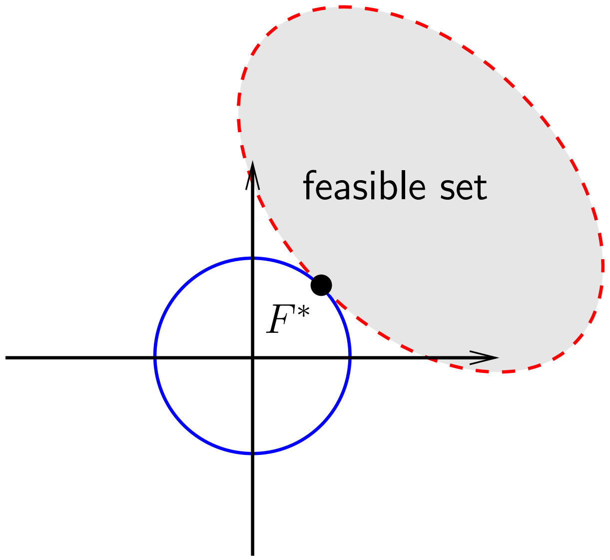
Definition. *The* convex envelope, f^{env}, *of a function* f *on a set* C *is the (point-wise) largest convex function that is an under estimator of* f *on* C.

example

$$\|x\|_1 = \sum_i |x_i| \text{ is convex envelope of } \mathrm{card}(x).$$

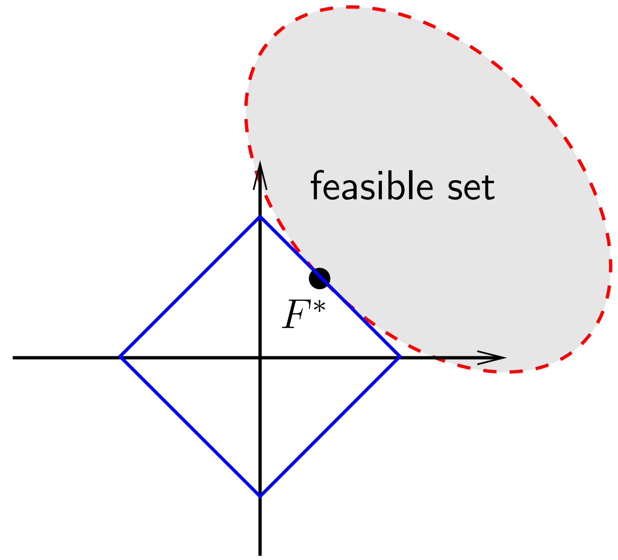


Sparsity Promoting Optimization



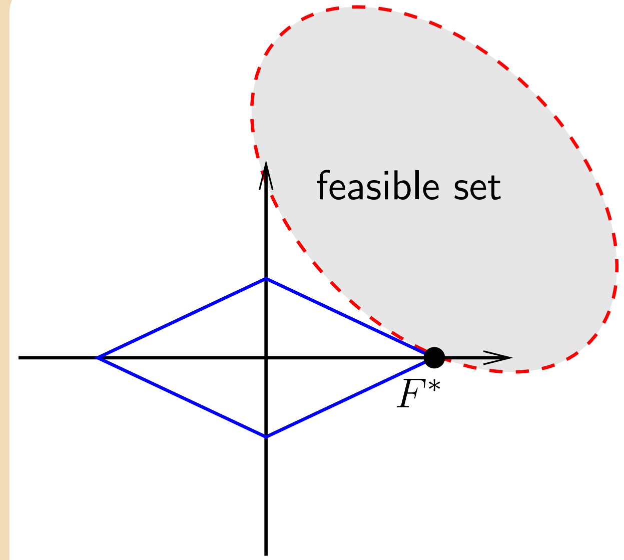
$$\min_{x \in \mathcal{X}} \|x\|_2$$

✳convex optimization
✳not sparse



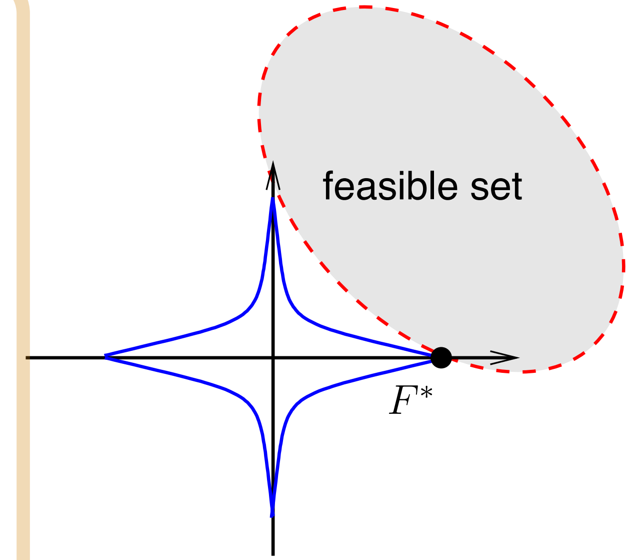
$$\min_{x \in \mathcal{X}} \|x\|_1$$

✳convex optimization
✳sparse for LP



$$\min_{x \in \mathcal{X}} \sum_i m_i |x_i|$$

✳convex optimization
✳sparse for SDP

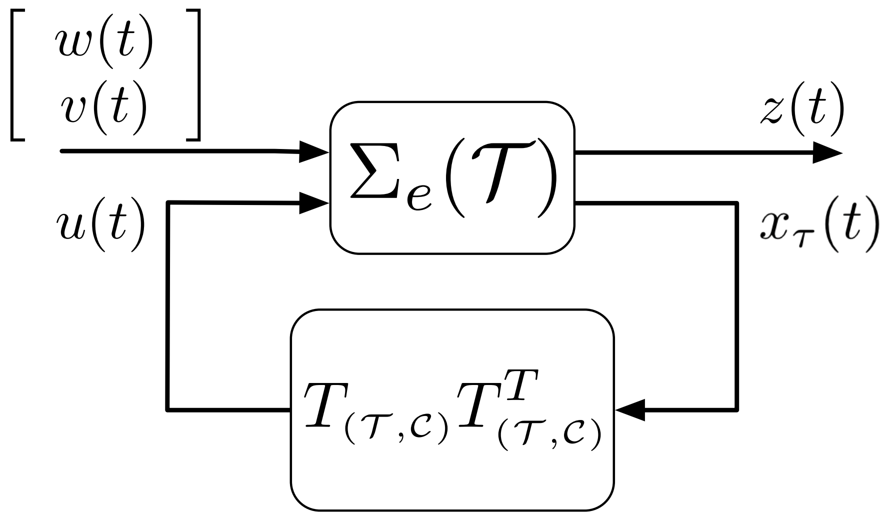


$$\min_{x \in \mathcal{X}} \|x\|_p$$

✳non-convex
✳sparse

re-weighted *l*-1 minimization algorithm
[Candes 2008]

Design of Cycles



$$\min_{T_{(\mathcal{T},\mathcal{C})} \in \mathbb{R}^{|\mathcal{V}|\times k}} \|\Sigma_e(\mathcal{G})\|_2,$$

Given a spanning tree, add **k** edges that maximize the performance improvement

$$\begin{aligned} \min_{M, w_i} \quad & \alpha \mathbf{trace}\left[M\right] + (1-\alpha)\sum_i m_i w_i \\ \text{s.t.} \quad & \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0 \\ & \sum_i w_i = k, \quad 0 \leq w_i \leq 1. \end{aligned}$$

Design of Cycles

Re-weighted *l*-1 minimization algorithm

① set counter $h = 0$

choose initial weights for each edge $\boxed{m_i^{(0)}}$ combinatorial insights used here!

② solve convex program - obtain optimal weights $w_i^{(h)}$

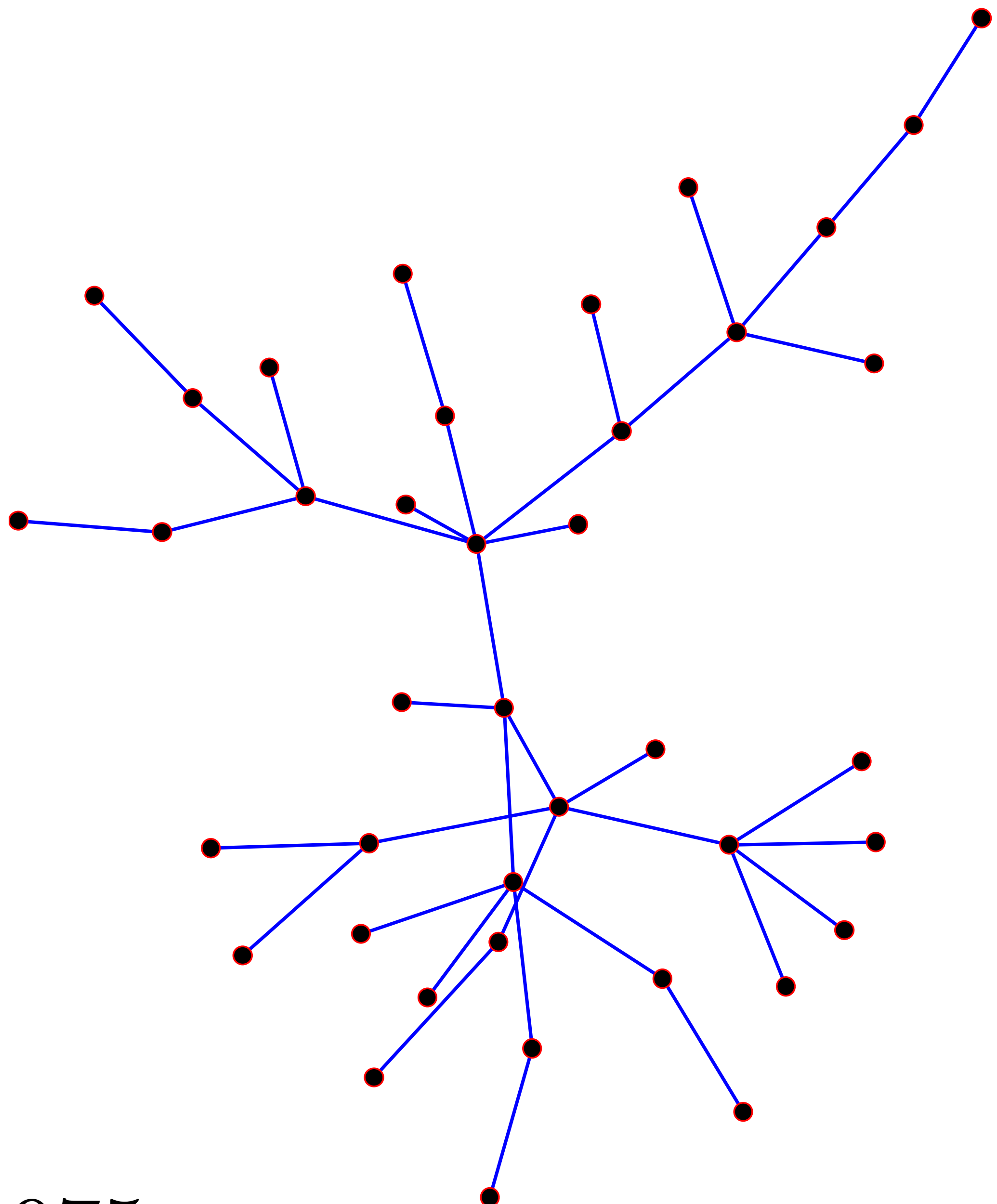
$$\begin{aligned} \min_{M, w_i} \quad & \alpha \mathbf{trace}\left[M\right] + (1-\alpha)\sum_i m_i^{(h)} w_i \\ \text{s.t.} \quad & \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0 \\ & \sum_i w_i = k, \quad 0 \leq w_i \leq 1. \end{aligned}$$

③ update weights $m_i^{(h+1)} = (w_i^{(h)} + \nu)^{-1}$

④ terminate on convergence, or
increment counter and go to step 2

[Candes 2008]

Simulation Examples



spanning tree
30 nodes

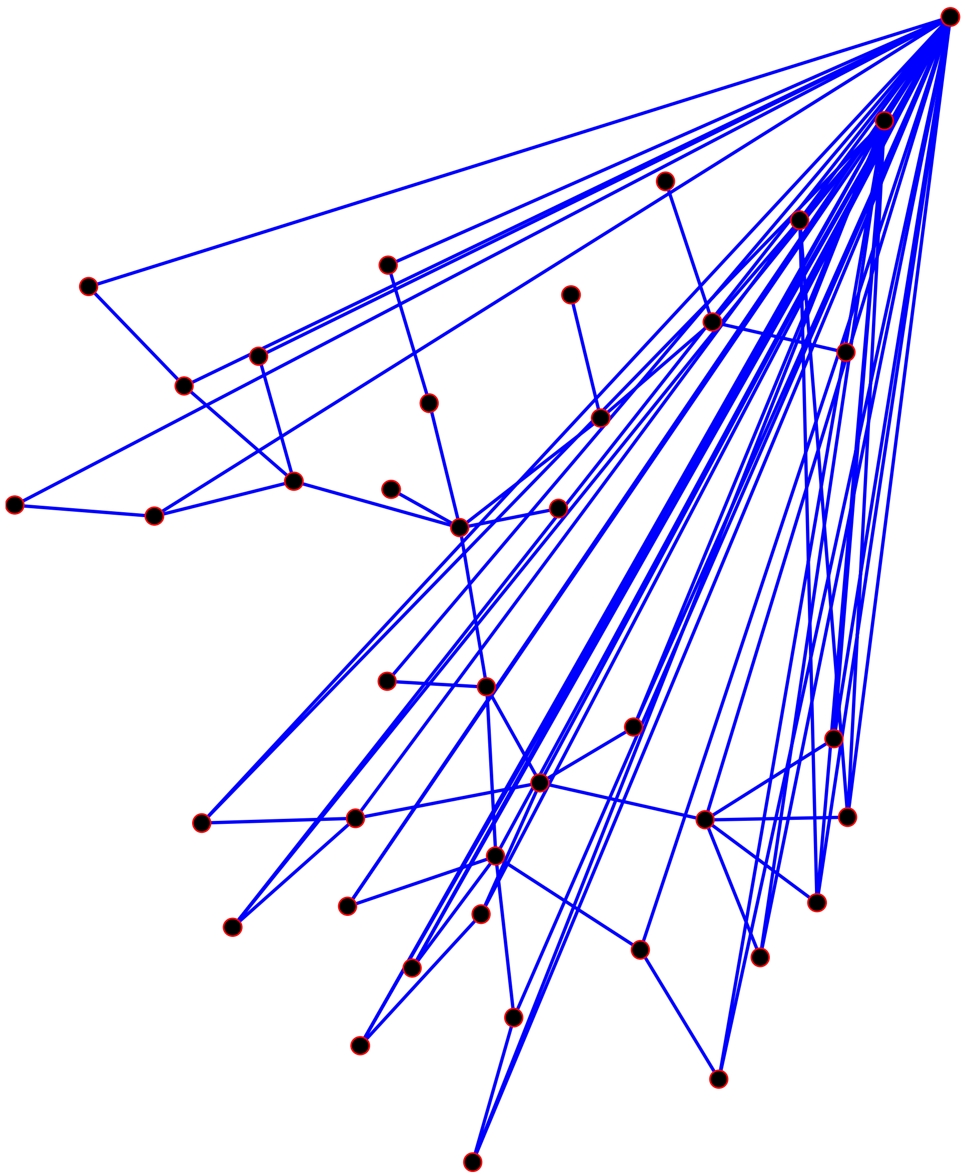
741 candidate edges

add 40 new edges

$$\|\Sigma(\mathcal{T})\|_2^2 = 58.5$$

$$\|\Sigma(K_n)\|_2^2 = 39.975$$

Simulation Examples



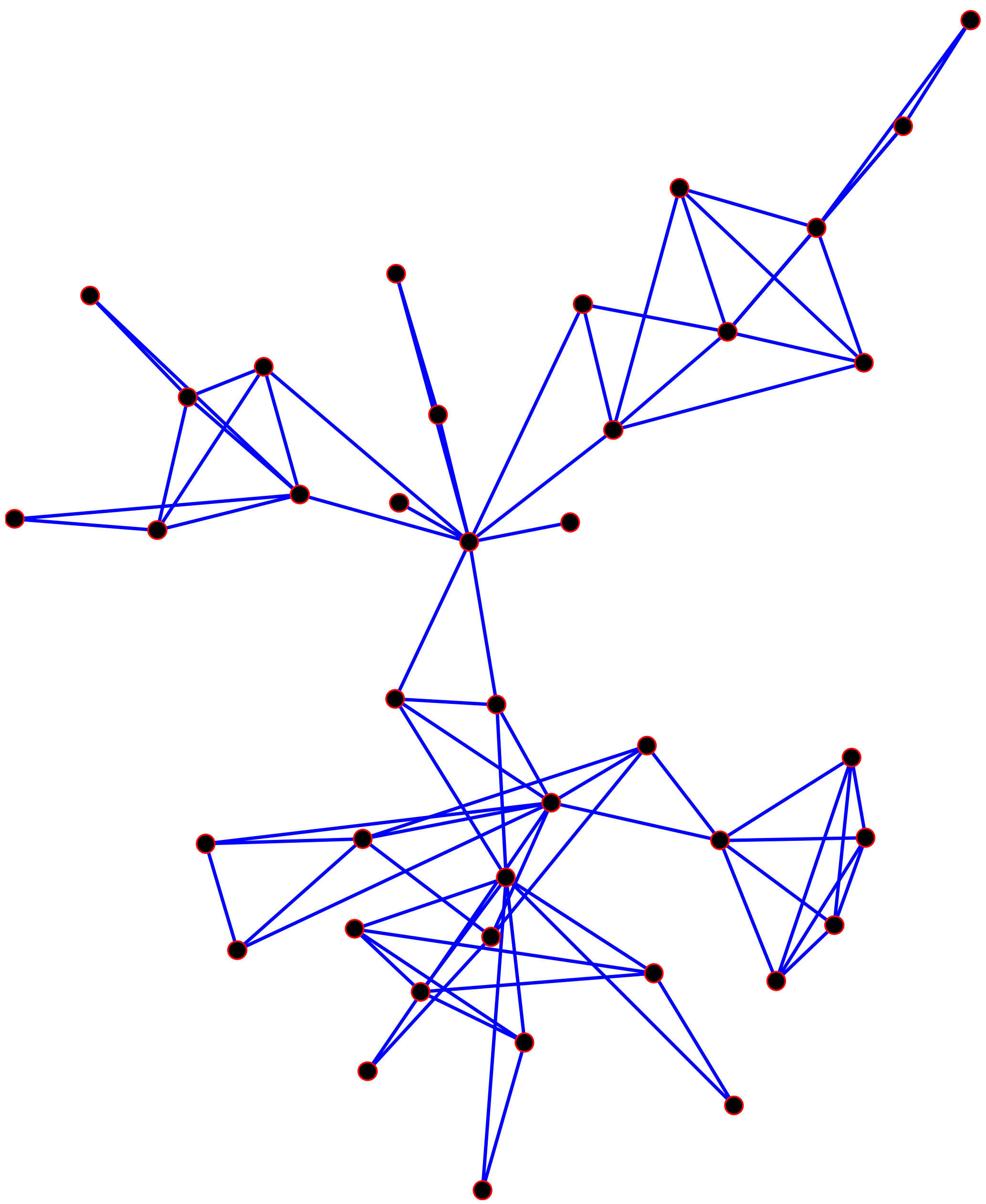
weights can be used to promote certain graph properties

"long cycle weights"

$$m_i = \mathbf{diam}(\mathcal{G}) - \|c_i\|_1 + 1$$

$$\|\Sigma(\mathcal{G})\|_2^2 = 50.233$$

Simulation Examples



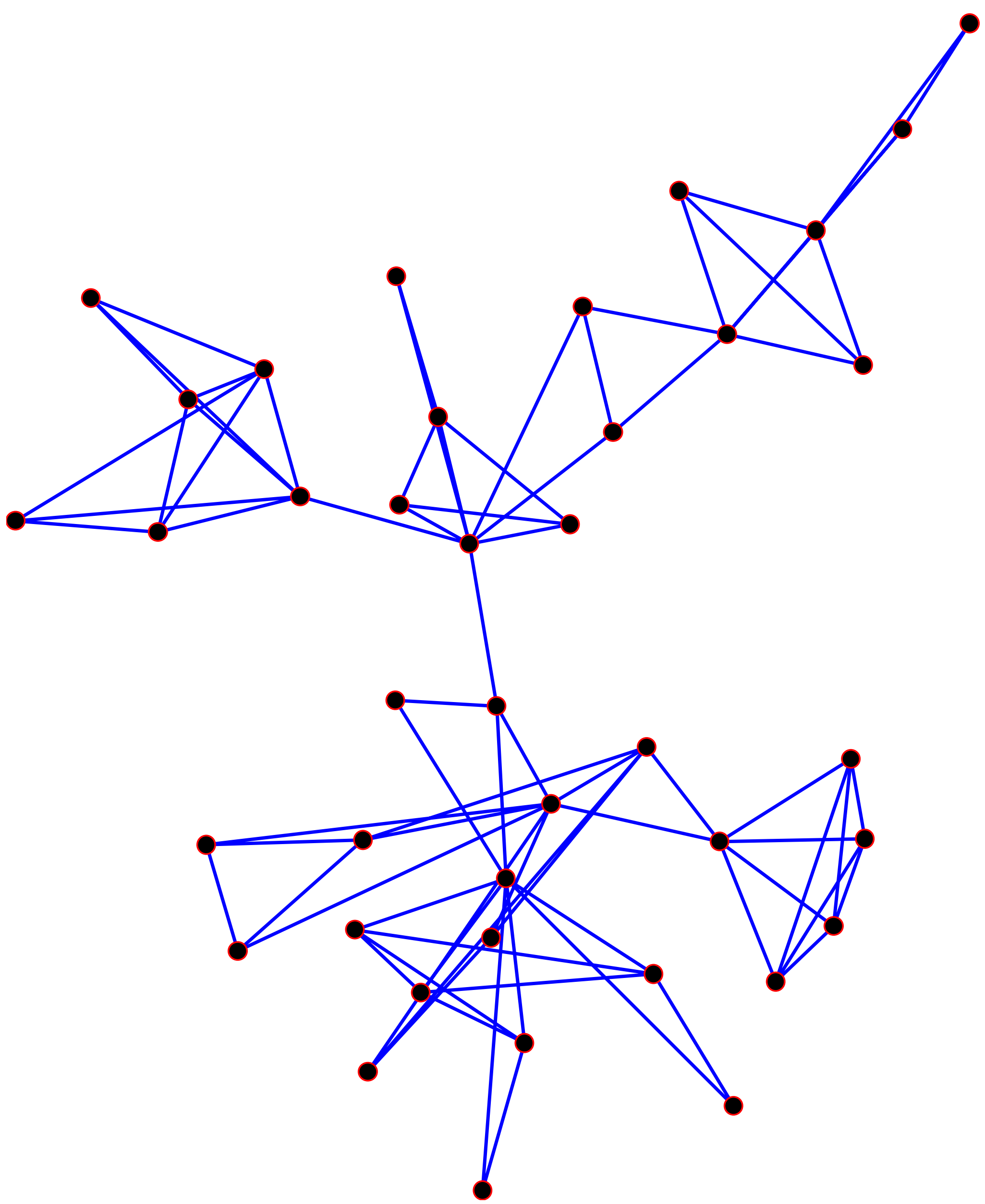
weights can be used to promote certain graph properties

"short cycle weights"

$$m_i = \|c_i\|_1$$

$$\|\Sigma(\mathcal{G})\|_2^2 = 48.704$$

Simulation Examples



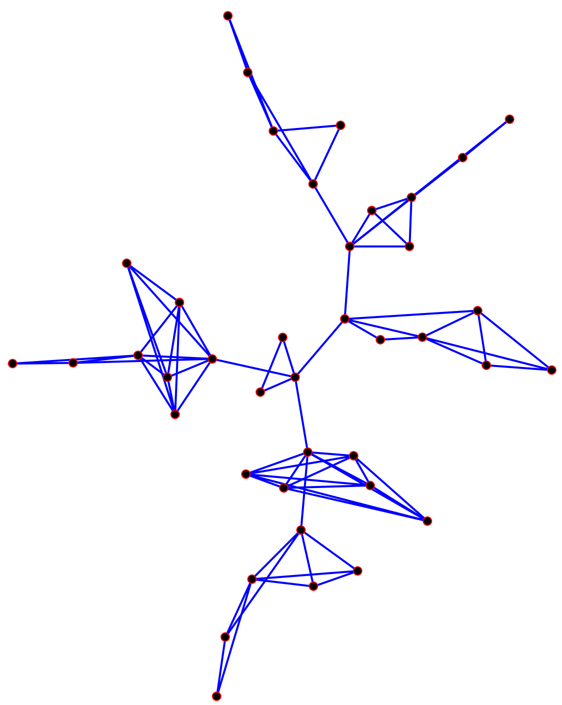
weights can be used to promote certain graph properties

"cycle correlation weights"

$$m_i = \frac{1}{|\mathcal{E}_c|} \sum_{j \neq i} \left| \left[T_{(\mathcal{T},\mathcal{C})} T_{(\mathcal{T},\mathcal{C})}^T \right]_{ij} \right|$$

$$\|\Sigma(\mathcal{G})\|_2^2 = 48.939$$

Simulation Examples



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