Analysis and Control of Multi-Agent Systems

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Formation Stabilization

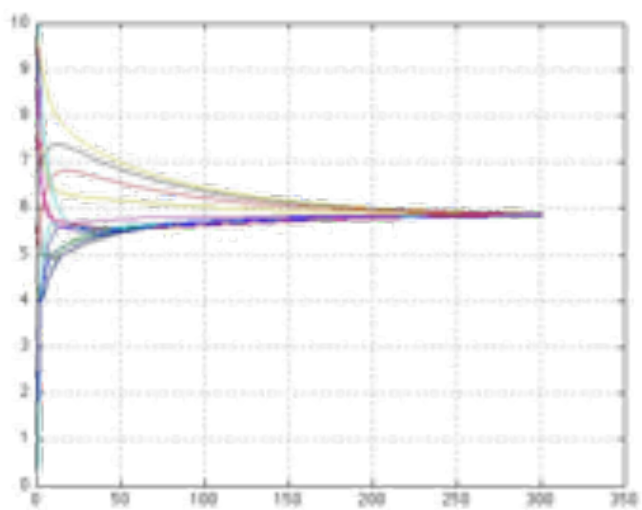
Consensus Feedback

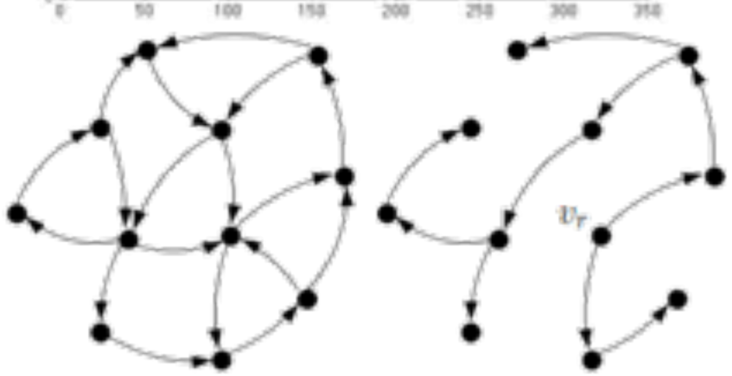
so far…

Linear Consensus

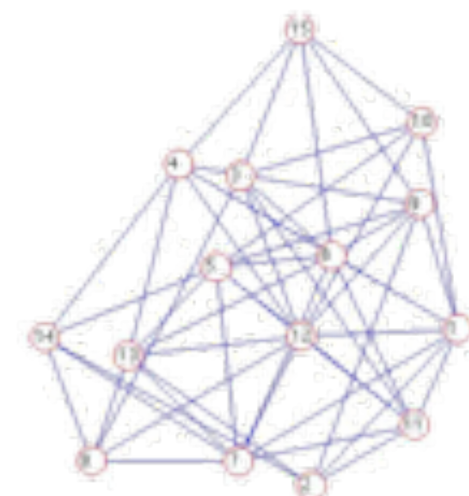
conditions for convergence to agreement

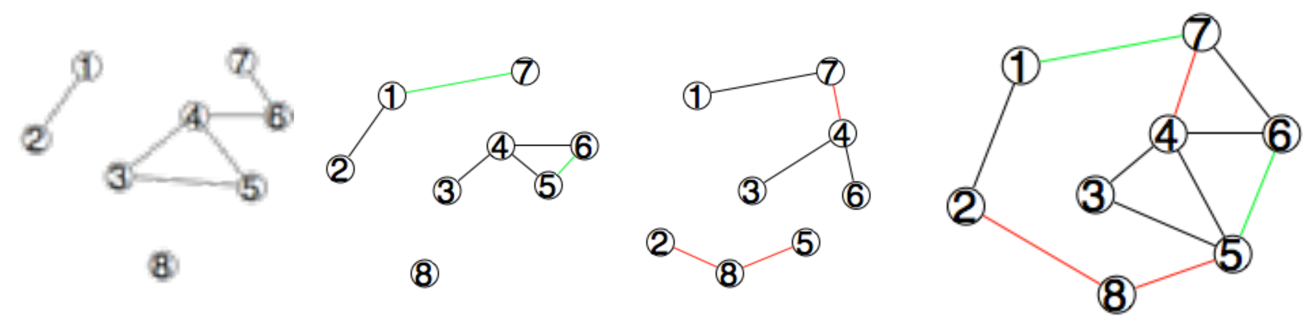
- connectedness
- rooted out-branching
- balanced and weakly connected
- jointly connected
- spectral conditions





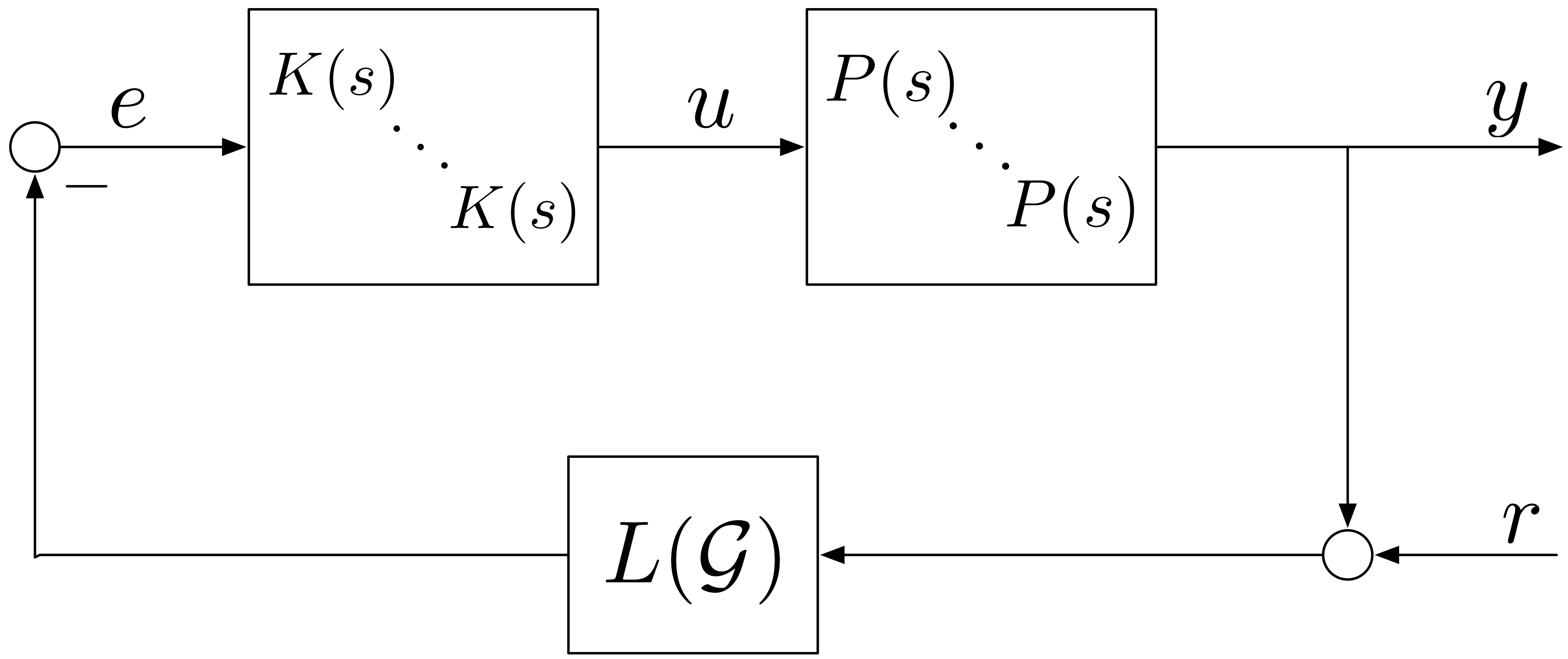
(a) Original Graph (b) Rooted Out-Branching Subgraph





Formation Stabilization

Consensus Feedback



Formation Stabilization

Assume identical (linear) dynamics for each agent

$$\Sigma_i \; : \; \begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t) \\ y_i(t) = C_1 x_i(t) \\ \quad \text{"internal" measurement} \\ z_i(t) = \frac{1}{d_i} \sum_{j \in \mathcal{N}(i)} C_2(x_i(t) - x_j(t)) \\ \quad \text{"network" measurement} \end{cases}$$

$$x_i \in \mathbb{R}^n \quad u_i \in \mathbb{R}^m \quad y_i \in \mathbb{R}^p$$

$$i = 1, \ldots, |\mathcal{V}| = N$$

fixed information exchange/sensor network

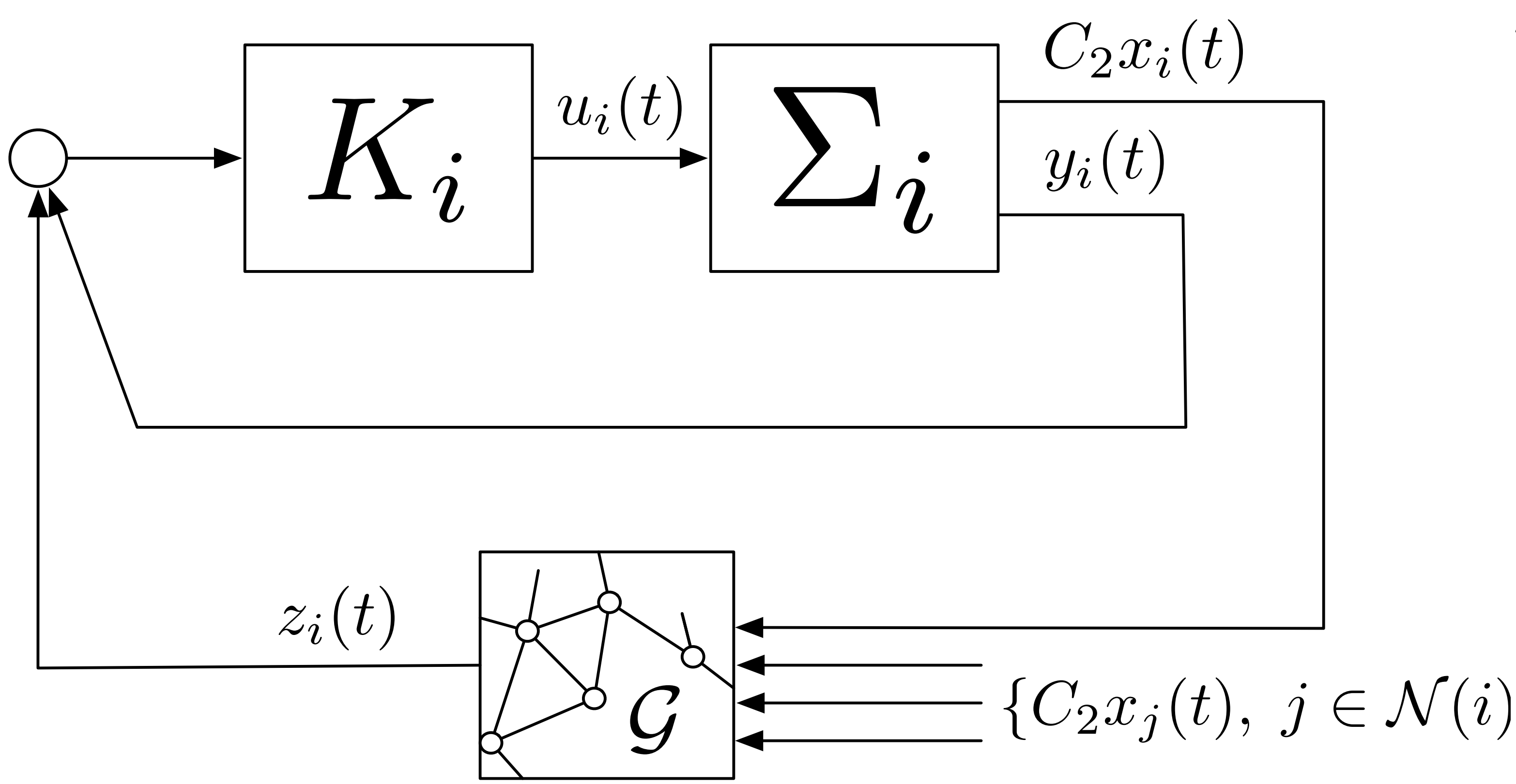
$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

Formation Stabilization

Assume a decentralized dynamic control law

$$K_i \ : \begin{cases} \dot{v}_i(t) &= A_K v_i(t) + B_{K1} y_i(t) + B_{K2} z_i(t) \\ u_i(t) &= C_K v_i(t) + D_{K1} y_i(t) + D_{K2} z_i(t) \end{cases}$$

$$i = 1, \ldots, |\mathcal{V}| = N$$



Formation Stabilization

$$\Sigma_i \ : \begin{cases} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= C_1 x_i(t) \\ z_i(t) &= \frac{1}{d_i}\sum_{j\in\mathcal{N}(i)} C_2(x_i(t) - x_j(t)) \end{cases}$$

$$K_i \ : \begin{cases} \dot{v}_i(t) &= A_K v_i(t) + B_{K1} y_i(t) + B_{K2} z_i(t) \\ u_i(t) &= C_K v_i(t) + D_{K1} y_i(t) + D_{K2} z_i(t) \end{cases}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix}$$

we need a "compact" way to write all of this!

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_N(t) \end{bmatrix} \quad \mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_N(t) \end{bmatrix} \quad \mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_N(t) \end{bmatrix}$$

Matrix Kronecker Product

$$A \in \mathbb{R}^{n \times m}$$

$$B \in \mathbb{R}^{p \times q}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np \times mq}$$

examples

$$\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} = I \otimes A$$

$$\mathbb{1}_n \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix Kronecker Product

$$A \in \mathbb{R}^{n\times m}$$

$$B \in \mathbb{R}^{p\times q}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}$$

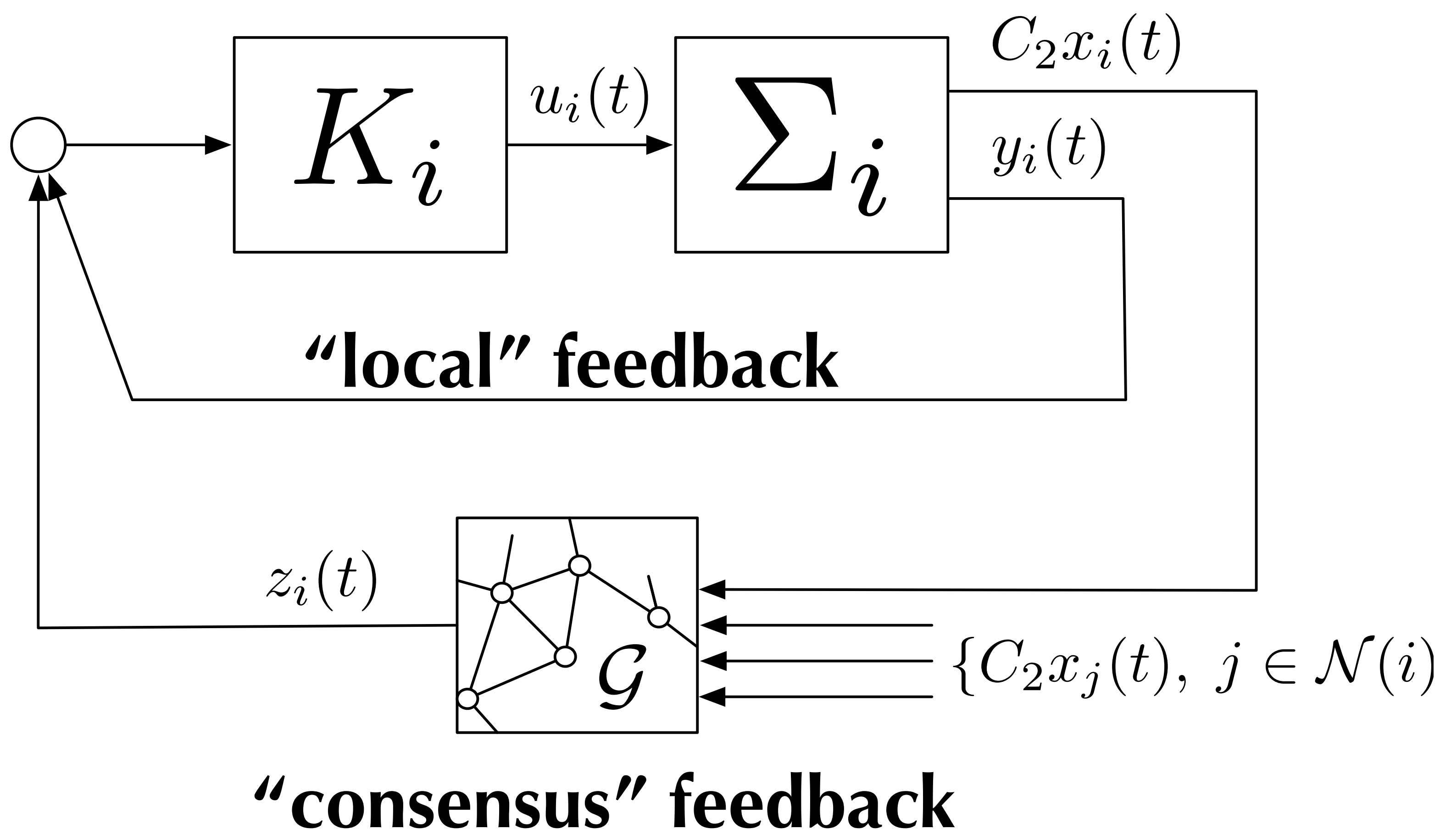
properties

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

$$A = U_A \Sigma_A V_A^T$$
$$B = U_B \Sigma_B V_B^T$$

$$(A \otimes B) = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)^T$$

Formation Stabilization



The closed-loop

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix}$$

Formation Stabilization

The closed-loop

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix}$$

$$\begin{aligned} A_{11} &= (I_N \otimes A + BD_{K1}C_1) + \left(\Delta^{-1}L(\mathcal{G}) \otimes BD_{K2}C_2\right) \\ A_{12} &= I_N \otimes BC_K \\ A_{21} &= (I_N \otimes B_{K1}C_1) + \left(\Delta^{-1}L(\mathcal{G}) \otimes B_{K2}C_2\right) \\ A_{22} &= I_N \otimes A_K \end{aligned}$$

The Normalized Laplacian

$$\tilde{L}(\mathcal{G}) = \Delta^{-1}(\mathcal{G})L(\mathcal{G})$$
$$= I - \Delta(\mathcal{G})^{-1}A(\mathcal{G})$$

Where are the eigenvalues located?

(Perron-Frobenius and Non-negative matrices)

Formation Stabilization

Theorem

A local controller K stabilizes the formation dynamics if and only if it simultaneously stabilizes the set of N systems

$$\begin{cases} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= C_1 x_i(t) \\ z_i(t) &= \lambda_i(\mathcal{G}) C_2 x_i(t), \end{cases}$$

where $\lambda_i(\mathcal{G})$ are the eigenvalues of $\tilde{L}(\mathcal{G})$.

Formation Stabilization

proof

coordinate transformation

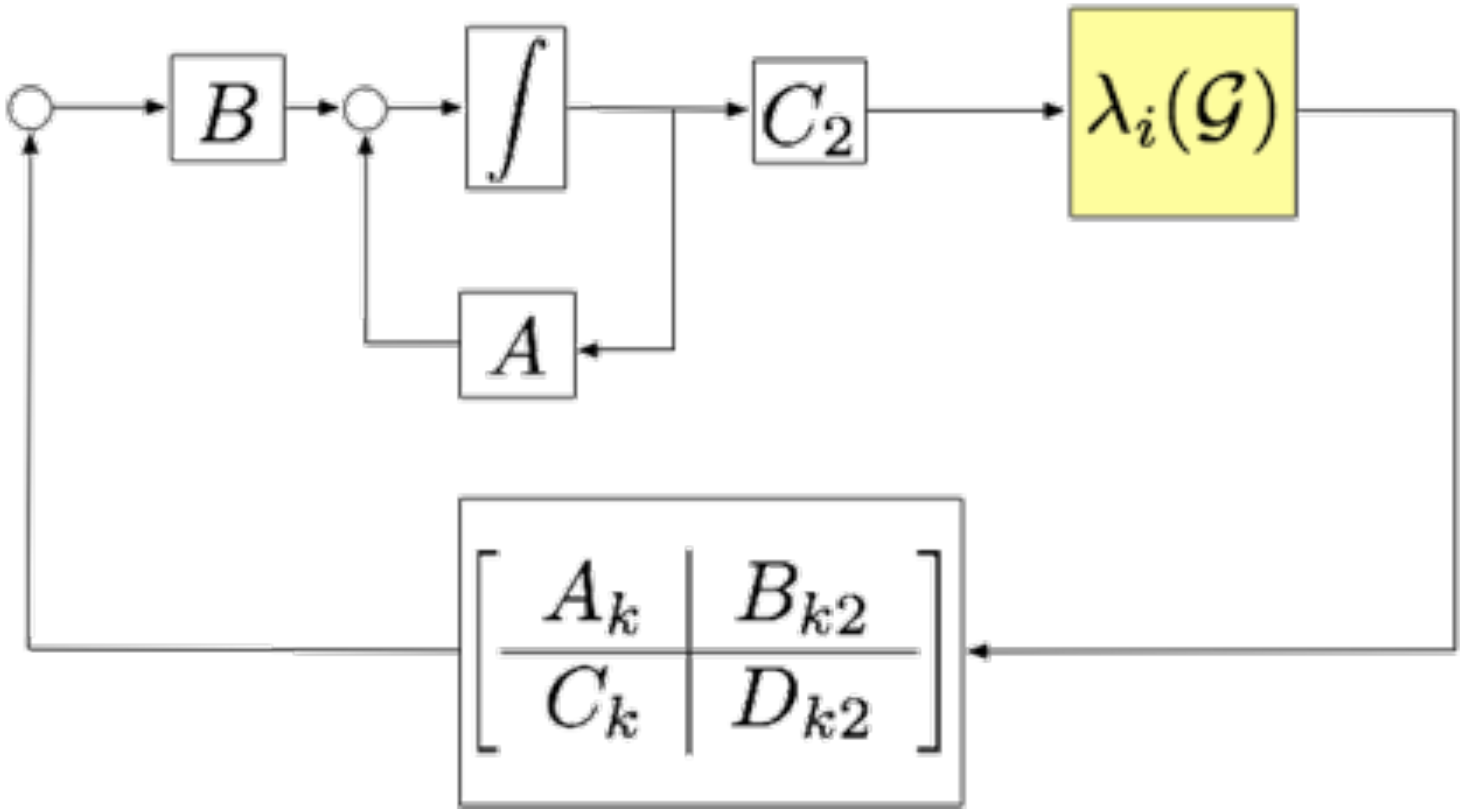
$$\begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} T^{-1} \otimes I_N & 0 \\ 0 & T^{-1} \otimes I_s \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$$

…leads to N decoupled sub-systems

$$\begin{bmatrix} \dot{\tilde{x}}_i(t) \\ \dot{\tilde{v}}_i(t) \end{bmatrix} = \begin{bmatrix} A + BD_{K1}C_1 + \lambda_i(\mathcal{G})BD_{K2}C_2 & BC_K \\ B_{K1}C_1 + \lambda_i(\mathcal{G})B_{K2}C_2 & A_K \end{bmatrix} \begin{bmatrix} \tilde{x}_i(t) \\ \tilde{v}_i(t) \end{bmatrix}$$

each subsystem must be stable!

Formation Stabilization

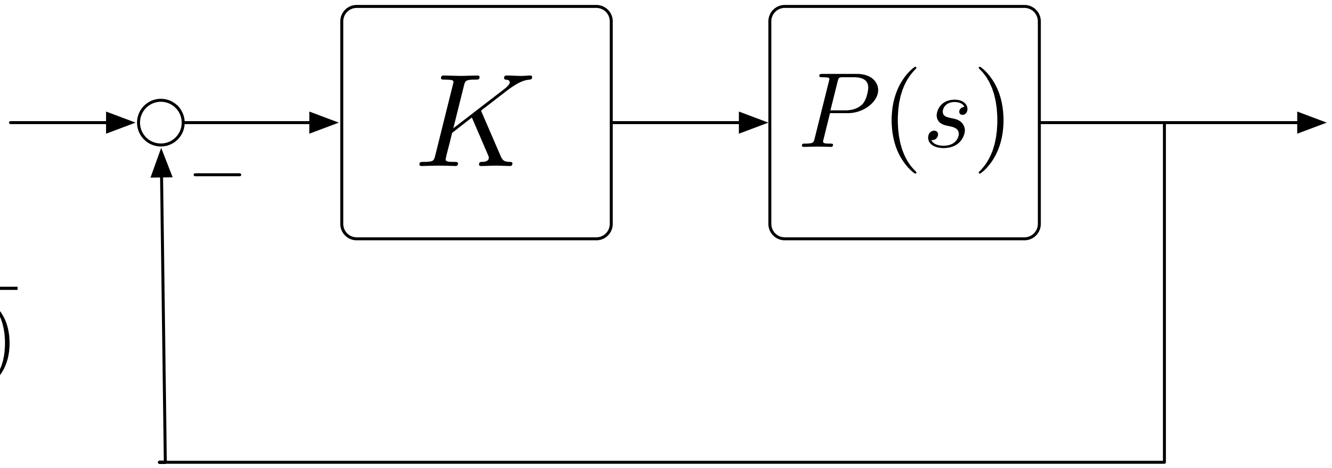


controllers should be "robust" against the eigenvalues of the Laplacian

Formation Stabilization

6 identical agents

$$P(s) = \frac{1}{s(1.4s+1)(0.33s+1)}$$



Proportional Control

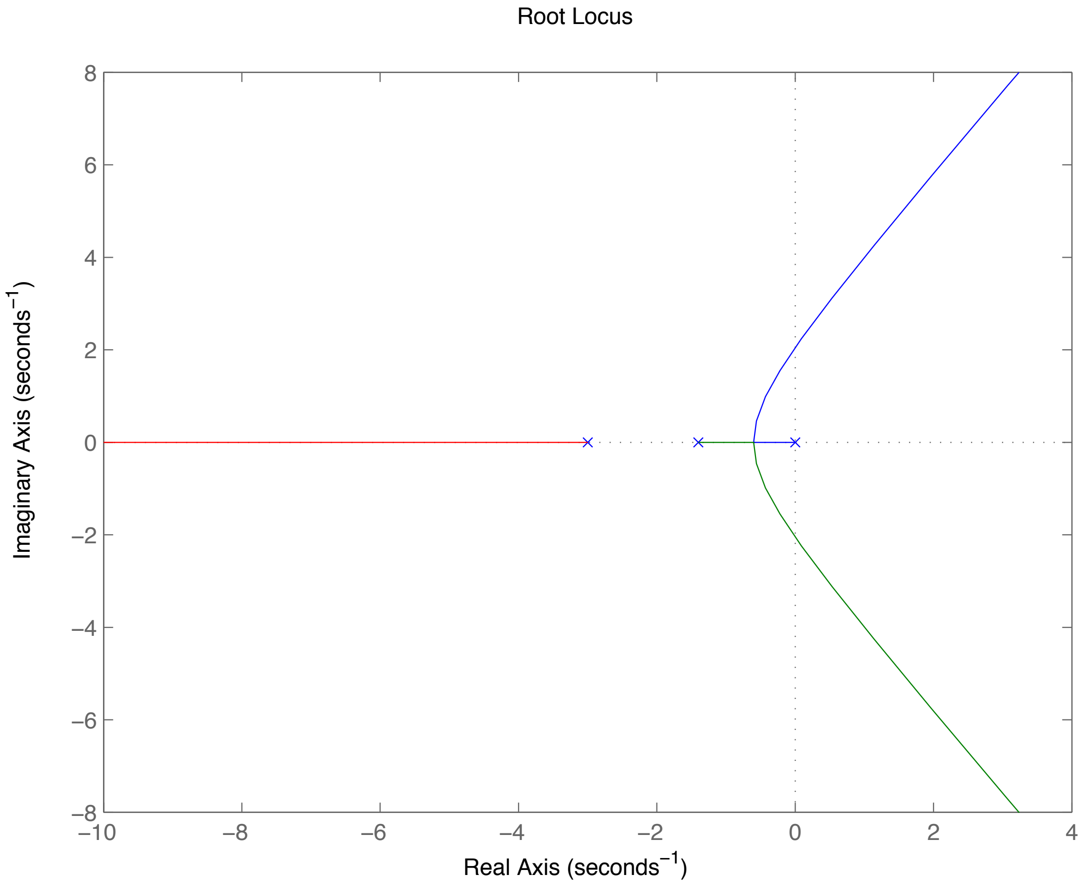
$$K = 1$$

stabilizable with proportional feedback

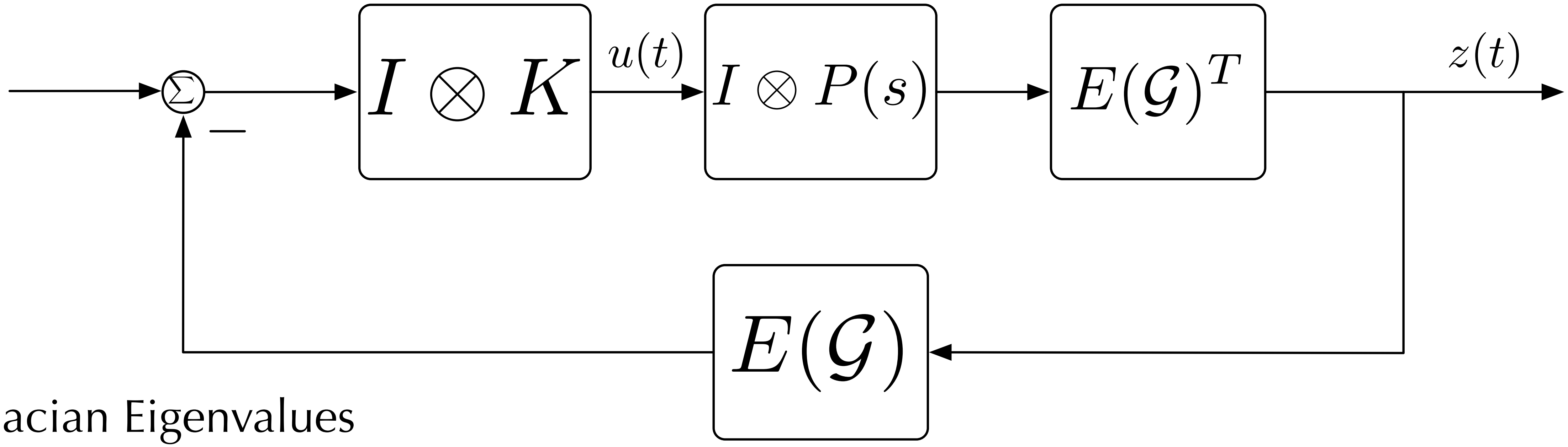
closed-loop eigenvalues

$$\{-3.55,\ -0.4249 \pm 1.001i\}$$

$$GM = 4.4$$



Formation Stabilization

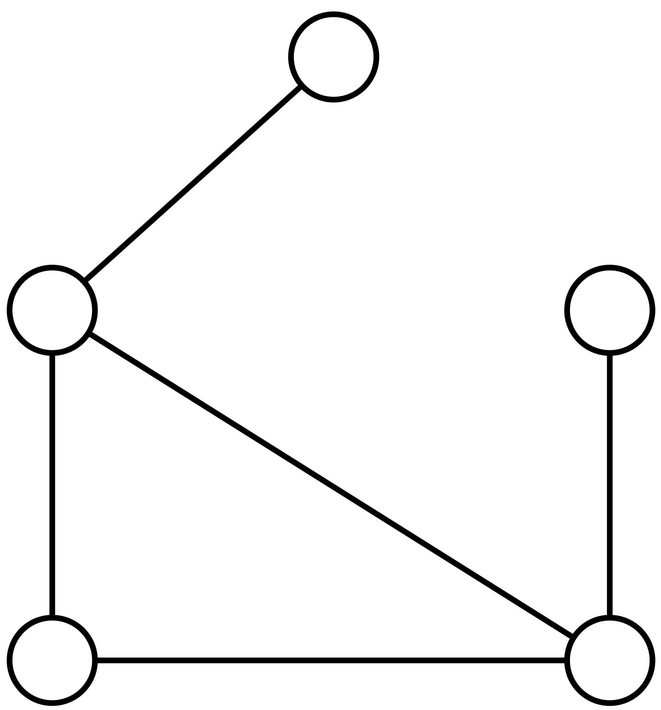


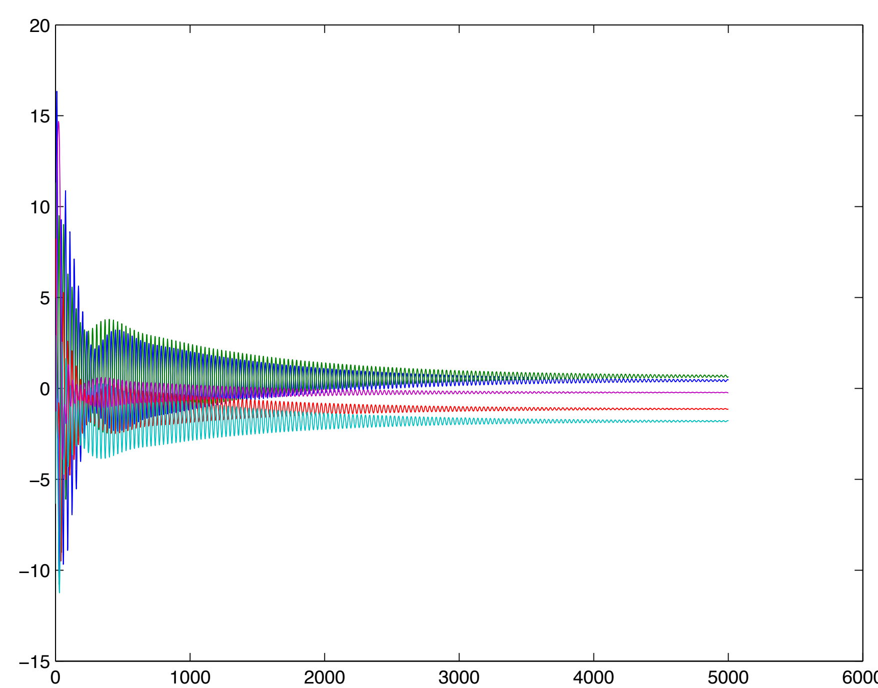
Laplacian Eigenvalues

$$\{0,\ 0.6972,\ 1.382,\ 3.618,\ 4.3028\}$$

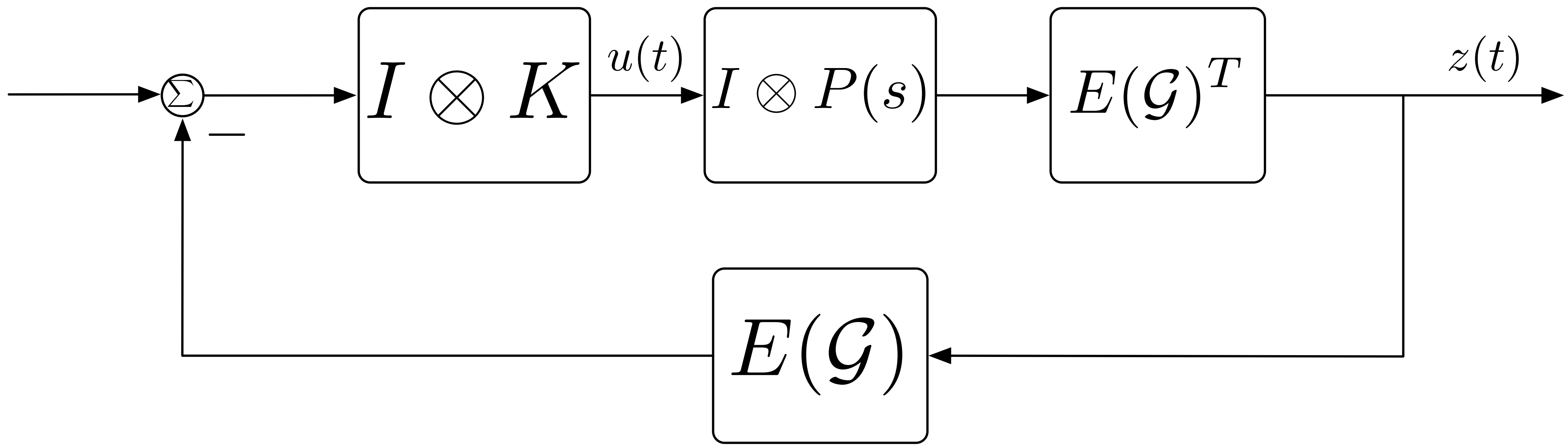
$$z(t) = (E(\mathcal{G})^T \otimes [1\,0\,0])x(t)$$

$$GM = 4.4$$





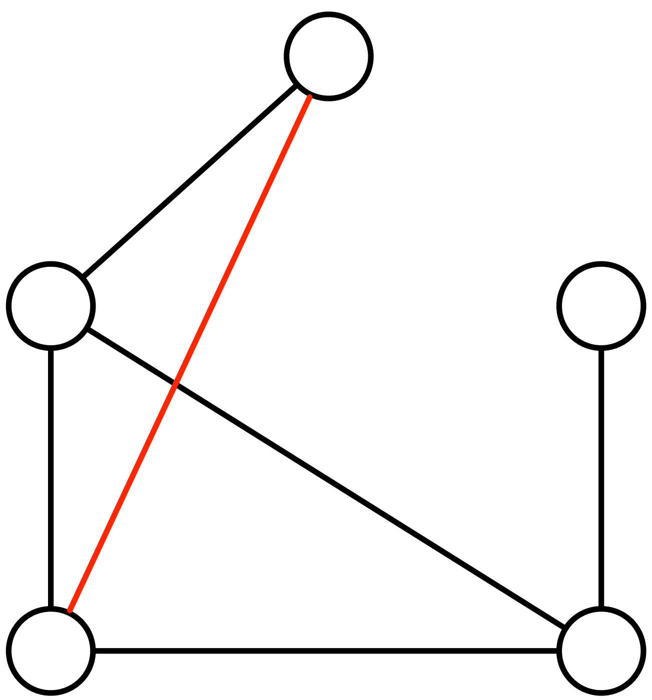
Formation Stabilization

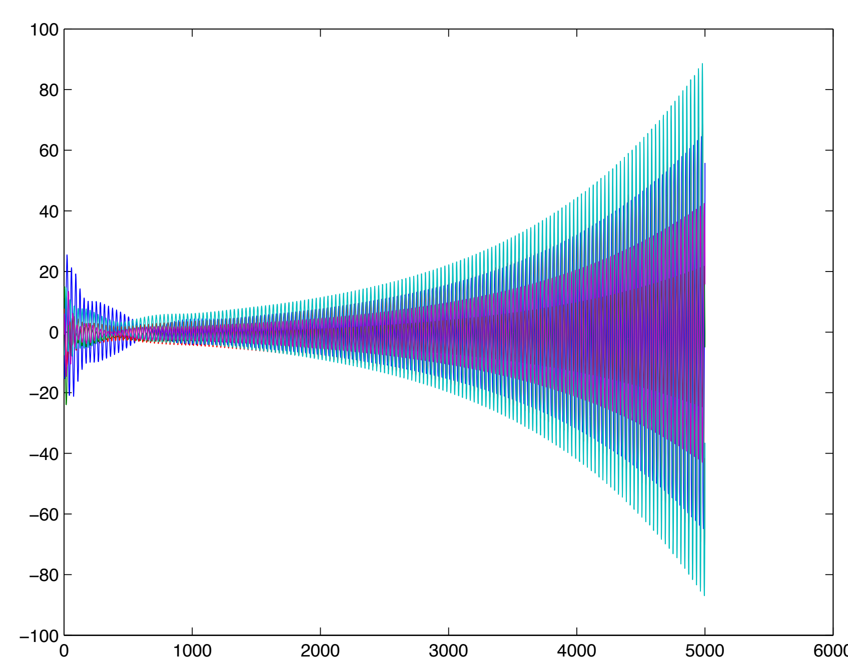


$$\{0,\ 0.8299,\ 2.6889,\ 4,\ 4.4812\}$$

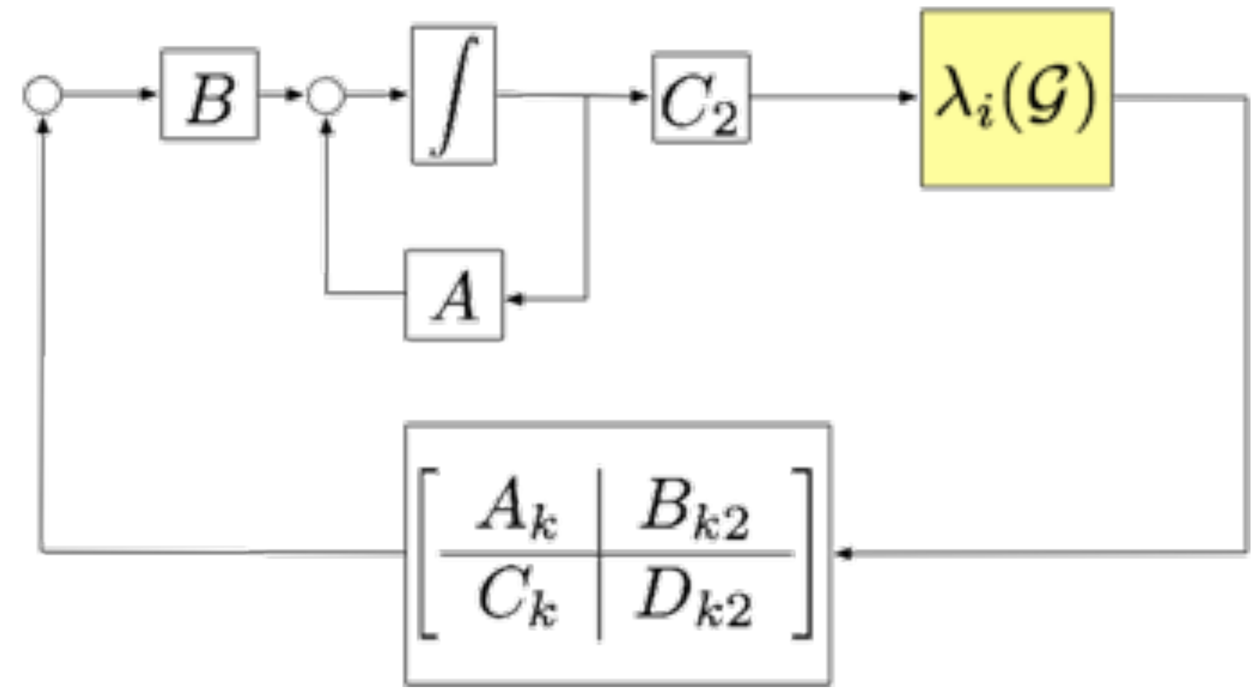
$$GM = 4.4$$

$$z(t) = (E(\mathcal{G})^T \otimes [1\,0\,0])x(t)$$





Formation Stabilization



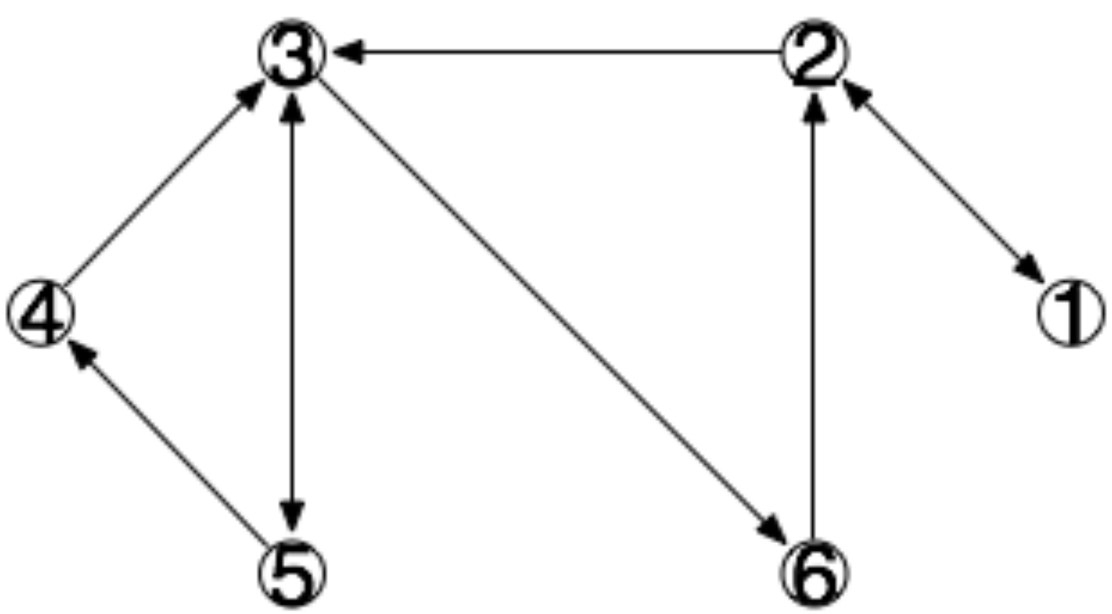
6 identical agents, double integrators with unit delay

$$P(s) = e^{-s}\frac{1}{s^2}$$

PD Control

$$K(s) = K_d s + K_p$$

communication graph

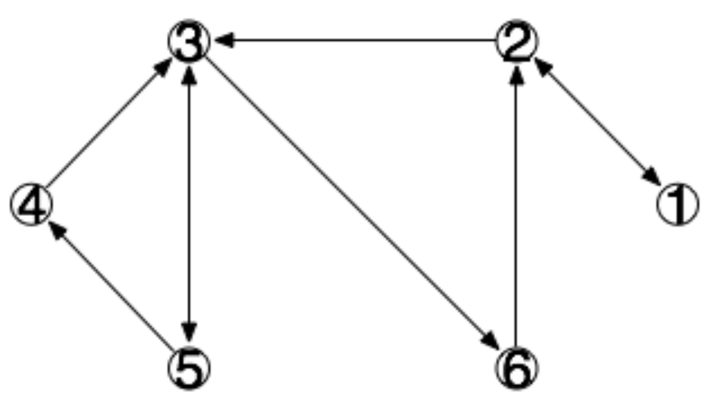


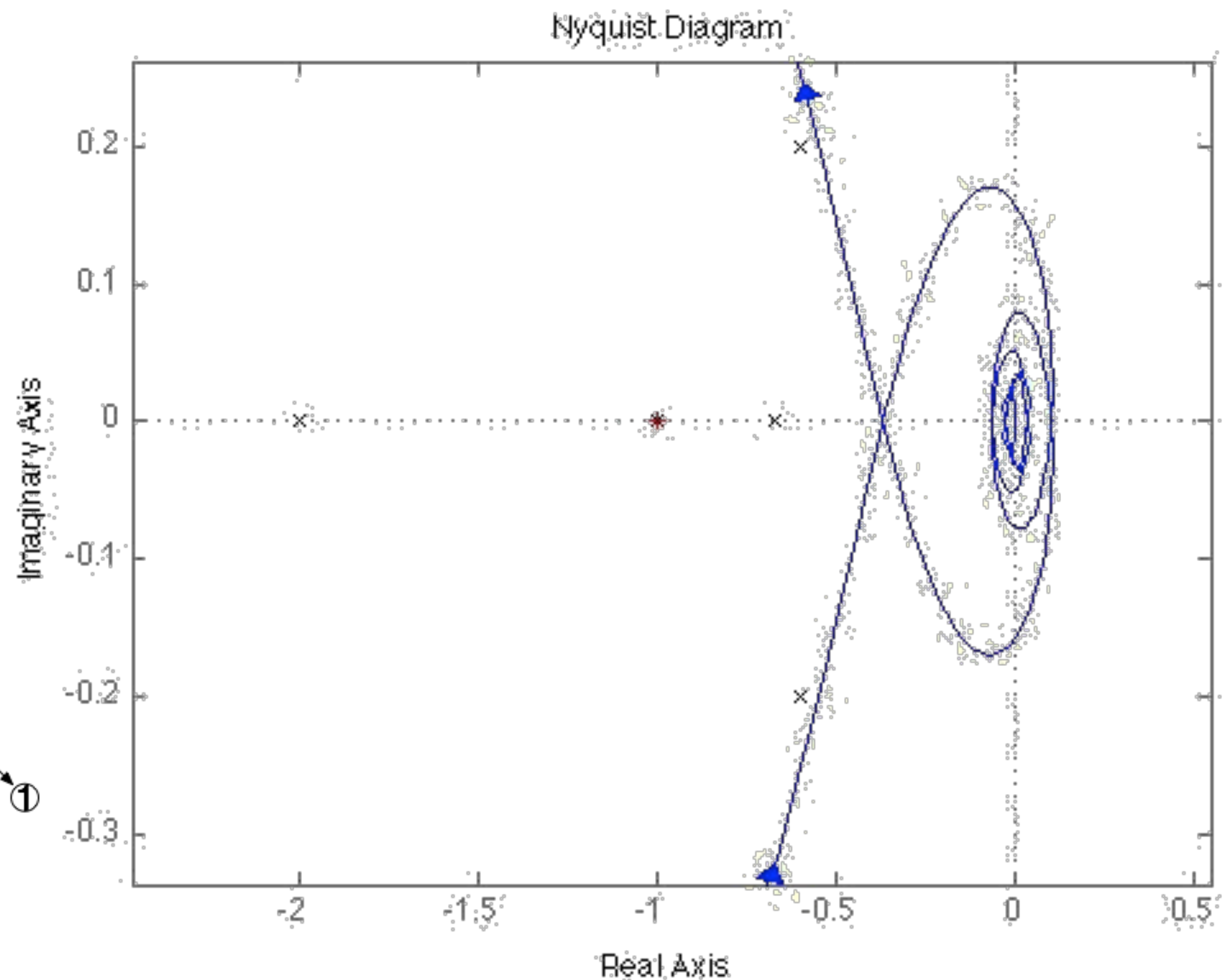
$$L(\mathcal{G}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Formation Stabilization

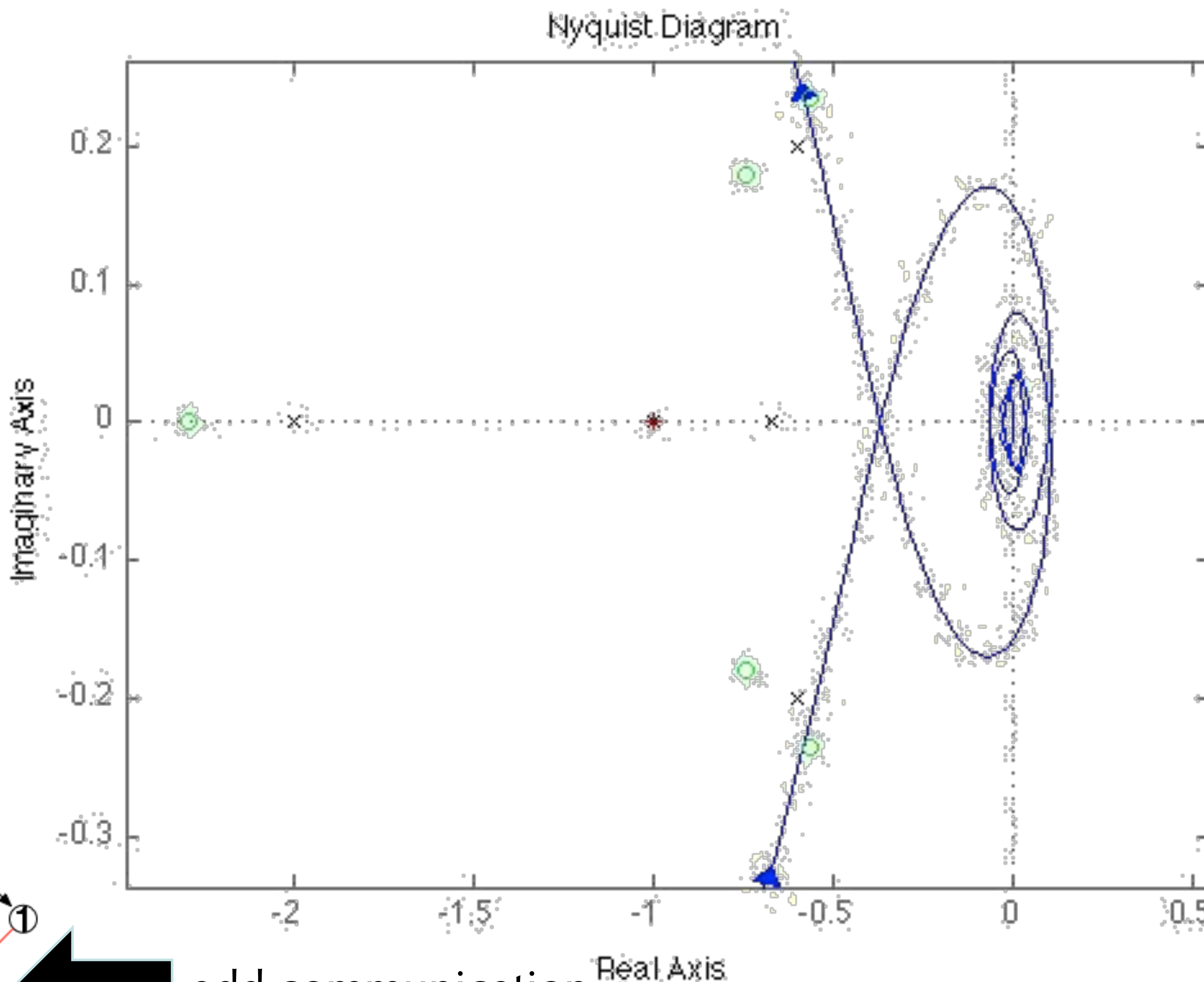
$$P(s) = e^{-s}\frac{1}{s^2}$$

$$K(s) = K_d s + K_p$$



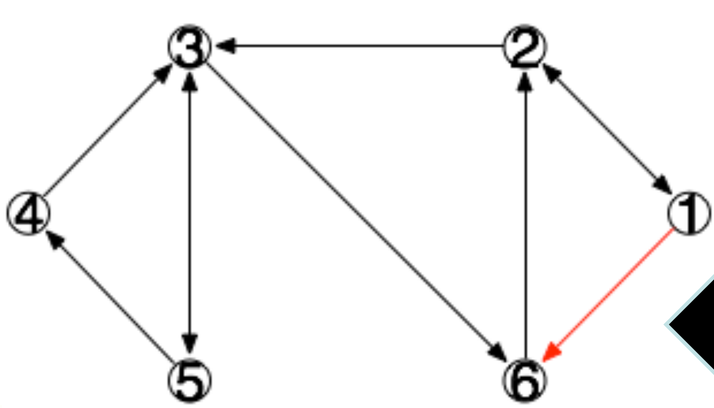


Formation Stabilization



$$P(s) = e^{s}\frac{1}{s^2}$$

$$K(s) = K_d s + K_p$$



add communication