Analysis and Control of Multi-Agent Systems

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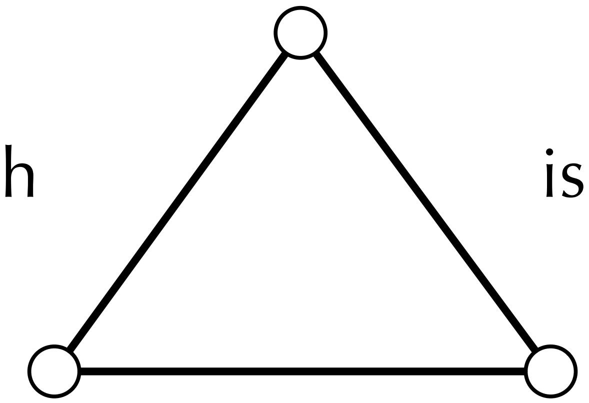
Graph Rigidity and Formation Control

Formation Control and Graph Rigidity

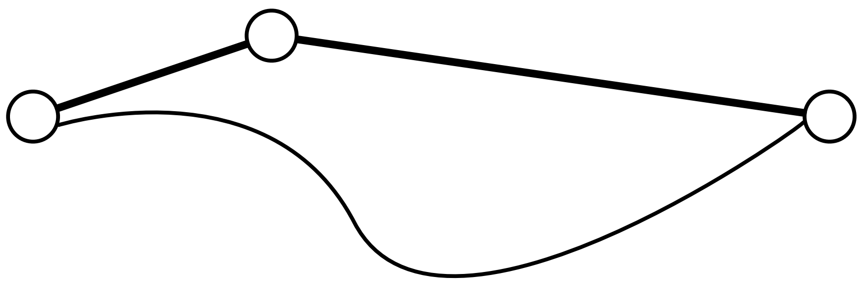
Graphs are a natural tool for describing formations!

but we must remember an important point…

this graph



is the same as this one…



the graphs we "draw" have no relation to "Euclidean space"

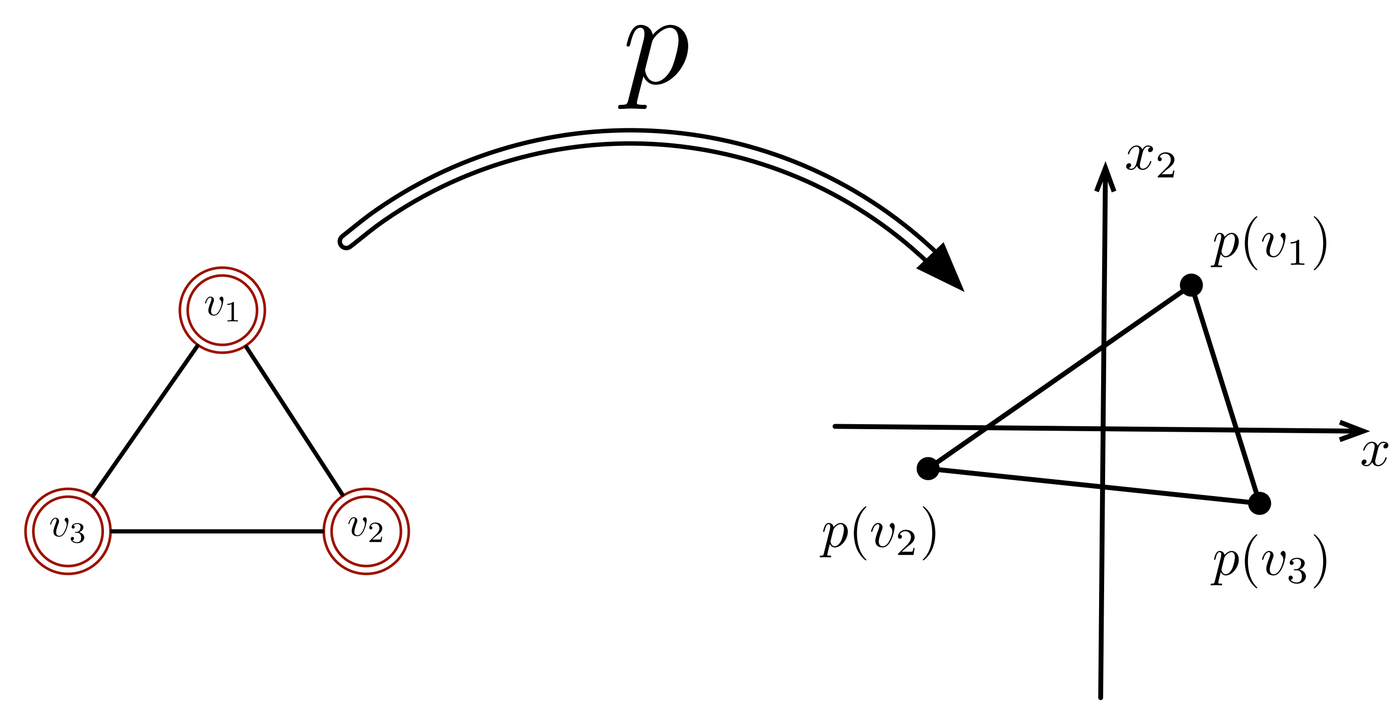
solution: formulate a language to "embed" a graph into Euclidean space

Bar-and-Joint Frameworks

A *framework* is a pair $(\mathcal{G}, p)$

$$\begin{cases} \mathcal{G} = (\mathcal{V}, \mathcal{E}) \\ p : \mathcal{V} \to \mathbb{R}^2 \end{cases}$$

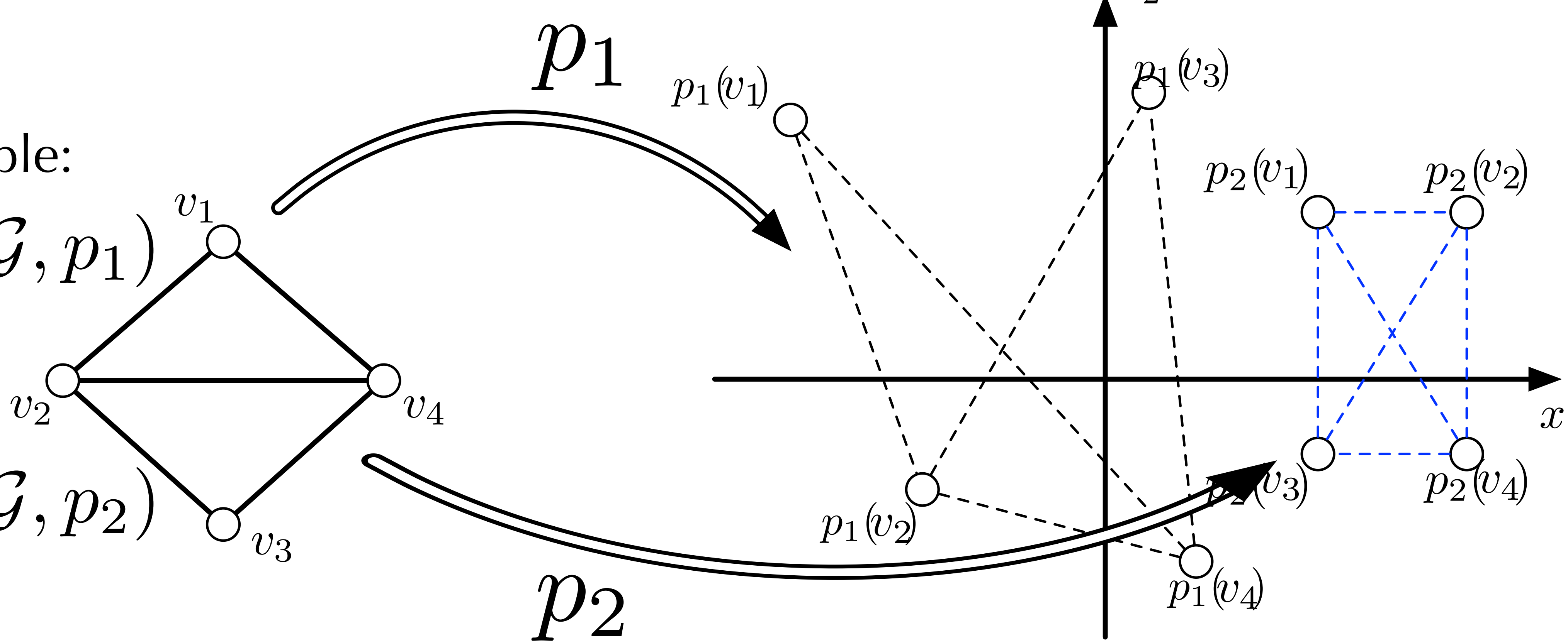
maps every vertex to a point in the plane



example:

$\mathcal{F}_1 = (\mathcal{G}, p_1)$

$\mathcal{F}_2 = (\mathcal{G}, p_2)$



Bar-and-Joint Frameworks

bar-and-joint frameworks

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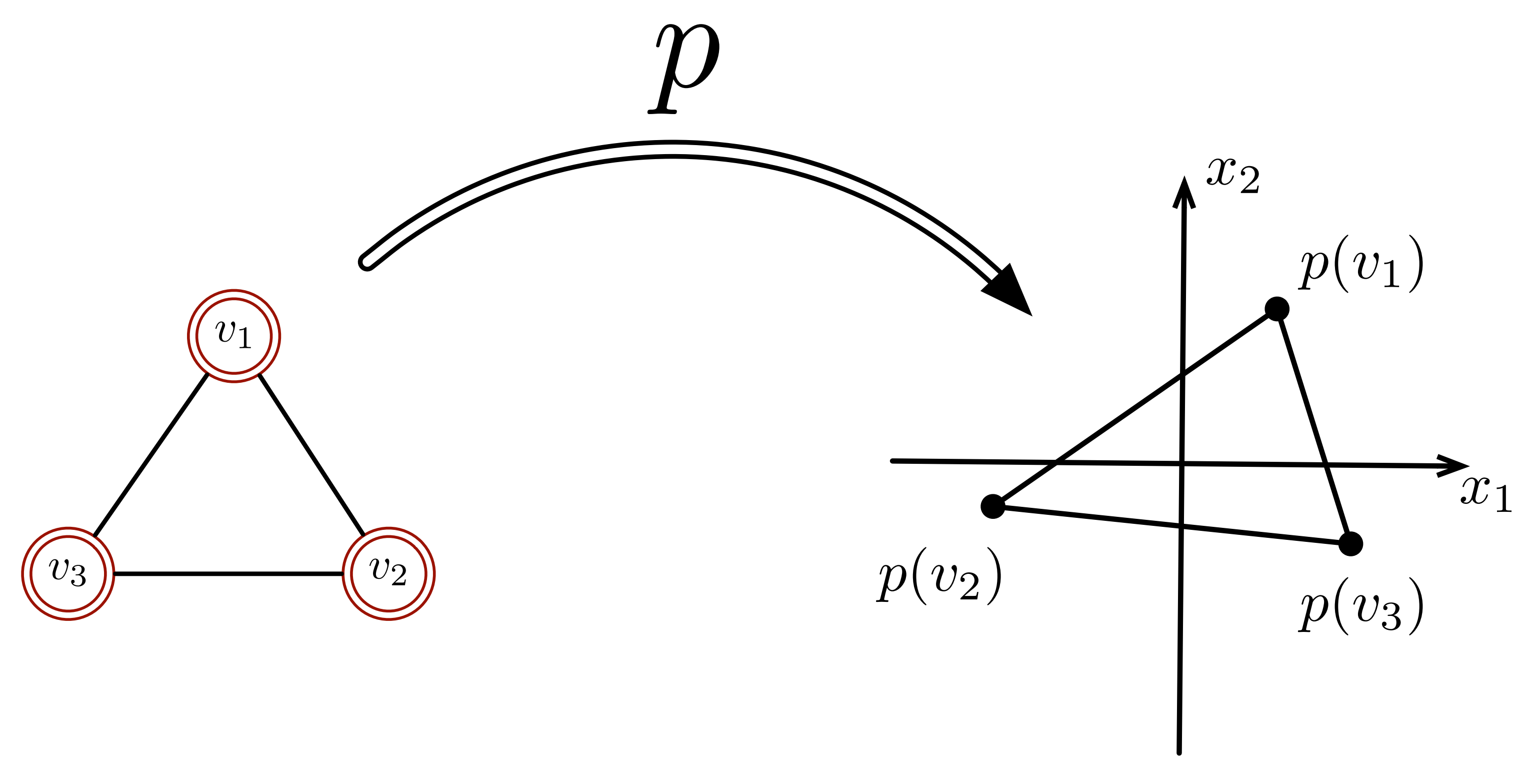


Two frameworks are *equivalent if*

$(\mathcal{G}, p_0)$ $(\mathcal{G}, p_1)$

$$\|p_0(v_i) - p_0(v_j)\| = \|p_1(v_i) - p_1(v_j)\|$$

$$\forall \{v_i, v_j\} \in \mathcal{E}$$

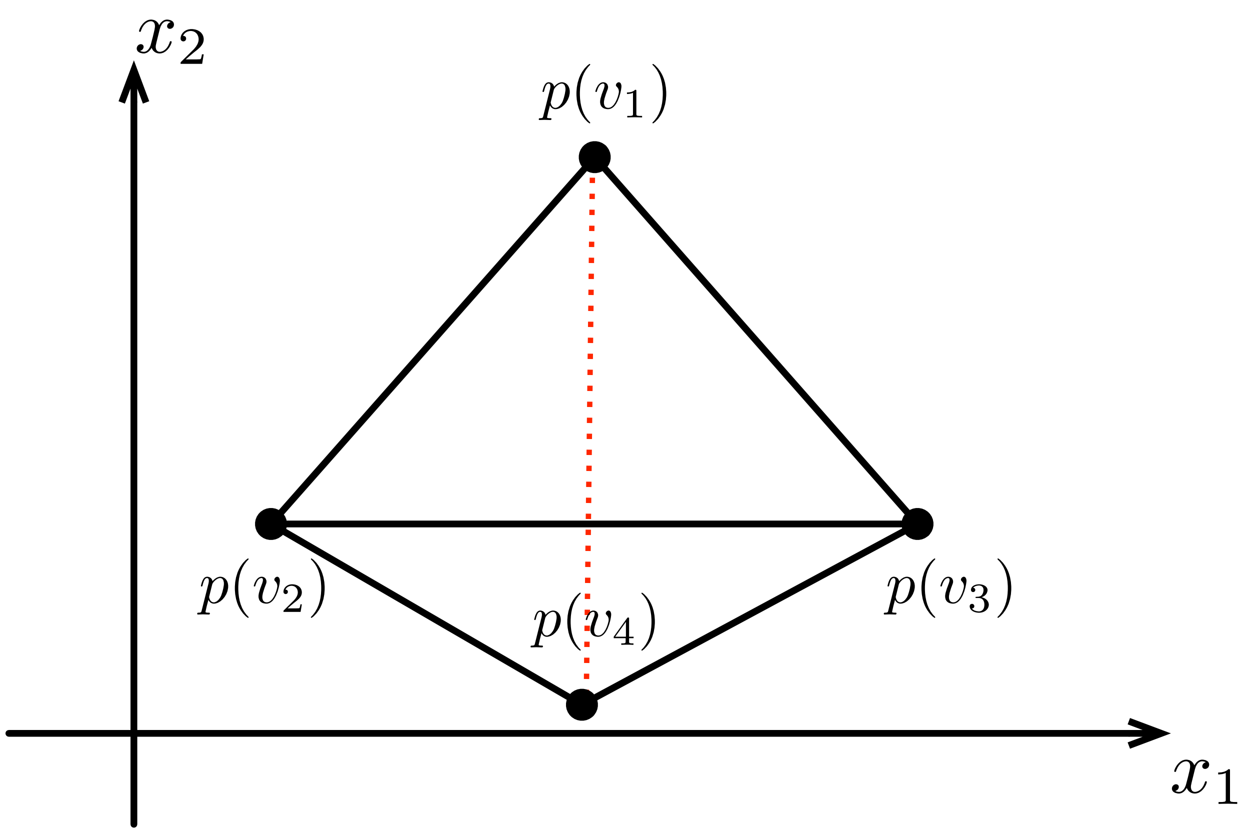
Two frameworks are *congruent if*

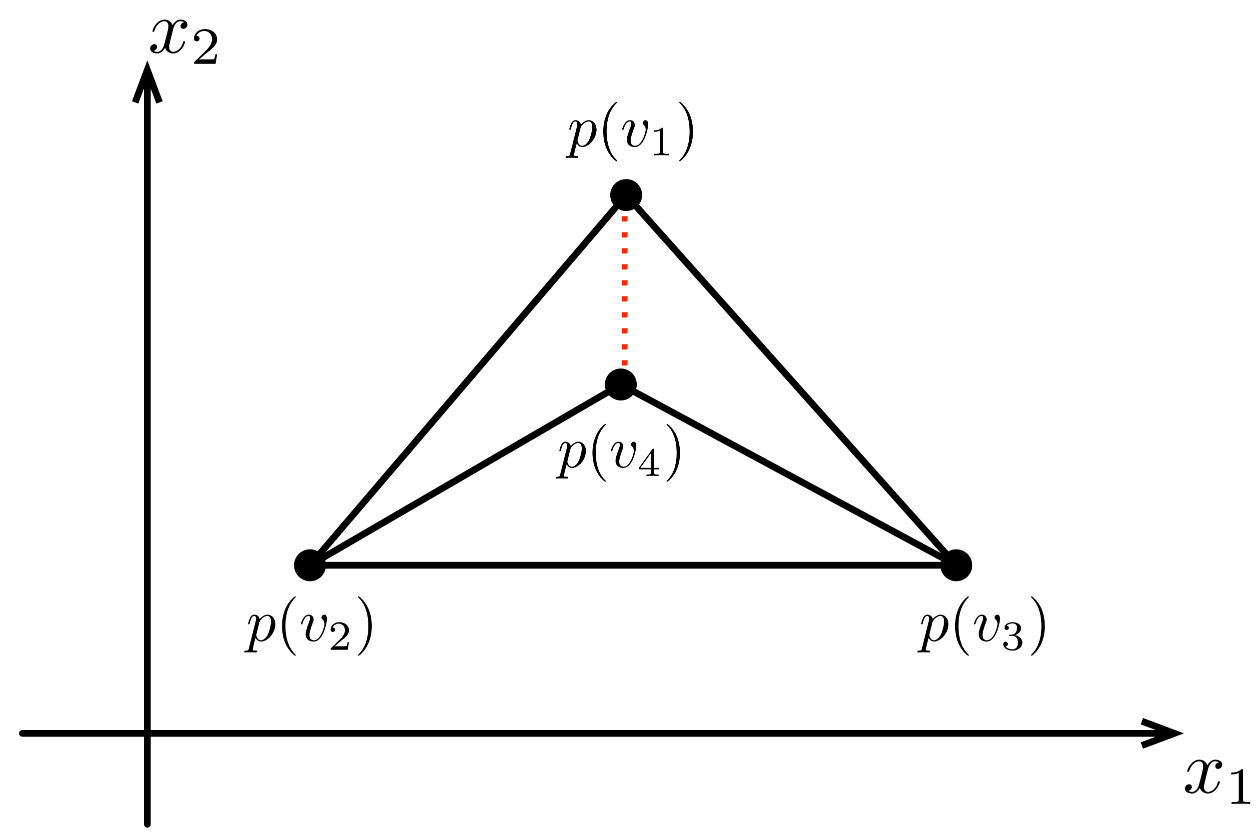
$(\mathcal{G}, p_0)$ $(\mathcal{G}, p_1)$

$$\|p_0(v_i) - p_0(v_j)\| = \|p_1(v_i) - p_1(v_j)\|$$

$$\forall v_i, v_j \in \mathcal{V}$$

Bar-and-Joint Frameworks





Frameworks are equivalent but not congruent!

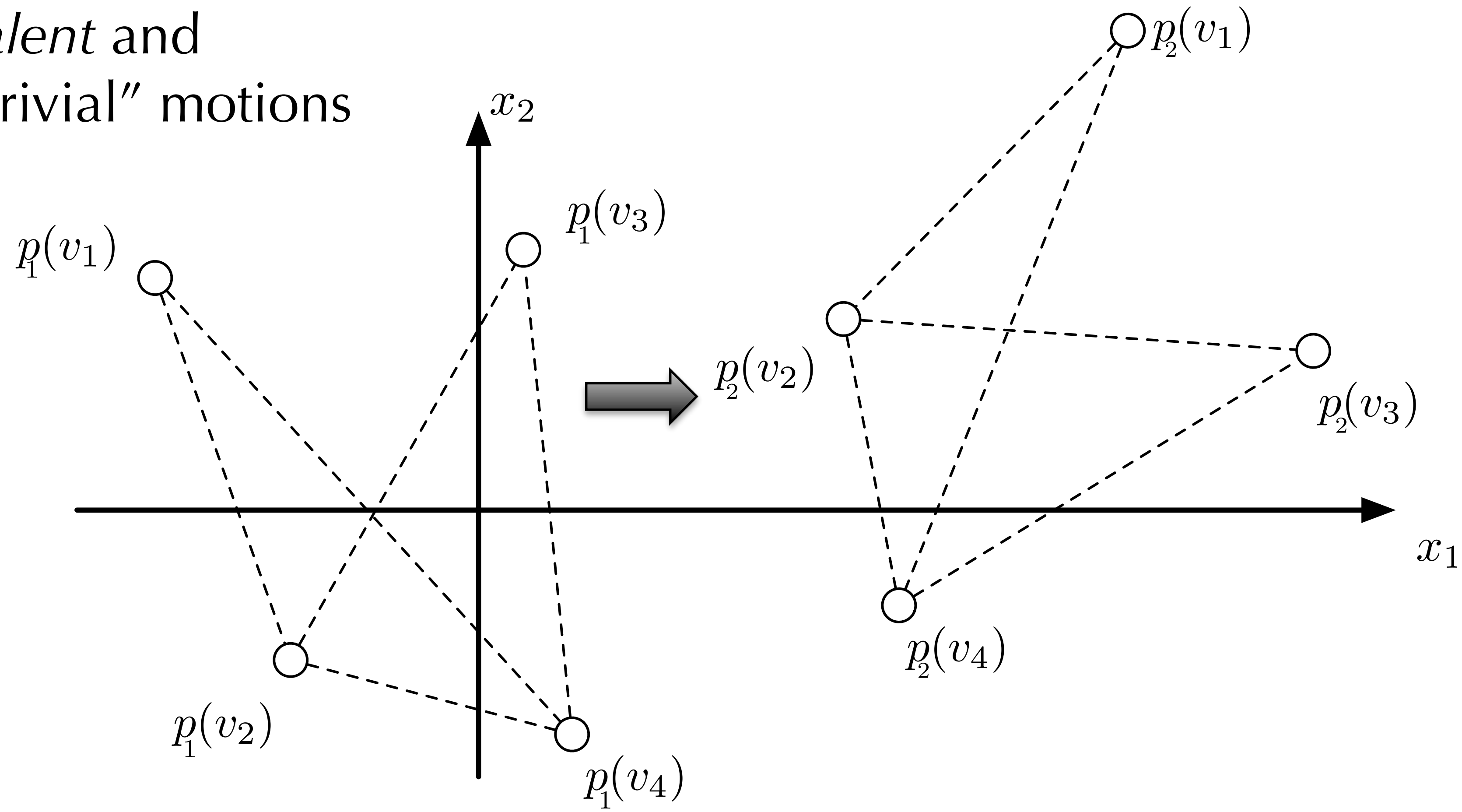
Bar-and-Joint Frameworks

Definition

A framework $(\mathcal{G}, p_0)$ is *globally rigid* if every framework that is equivalent to $(\mathcal{G}, p_0)$ is congruent to $(\mathcal{G}, p_0)$.

frameworks that are both *equivalent* and *congruent* are related by only "trivial" motions

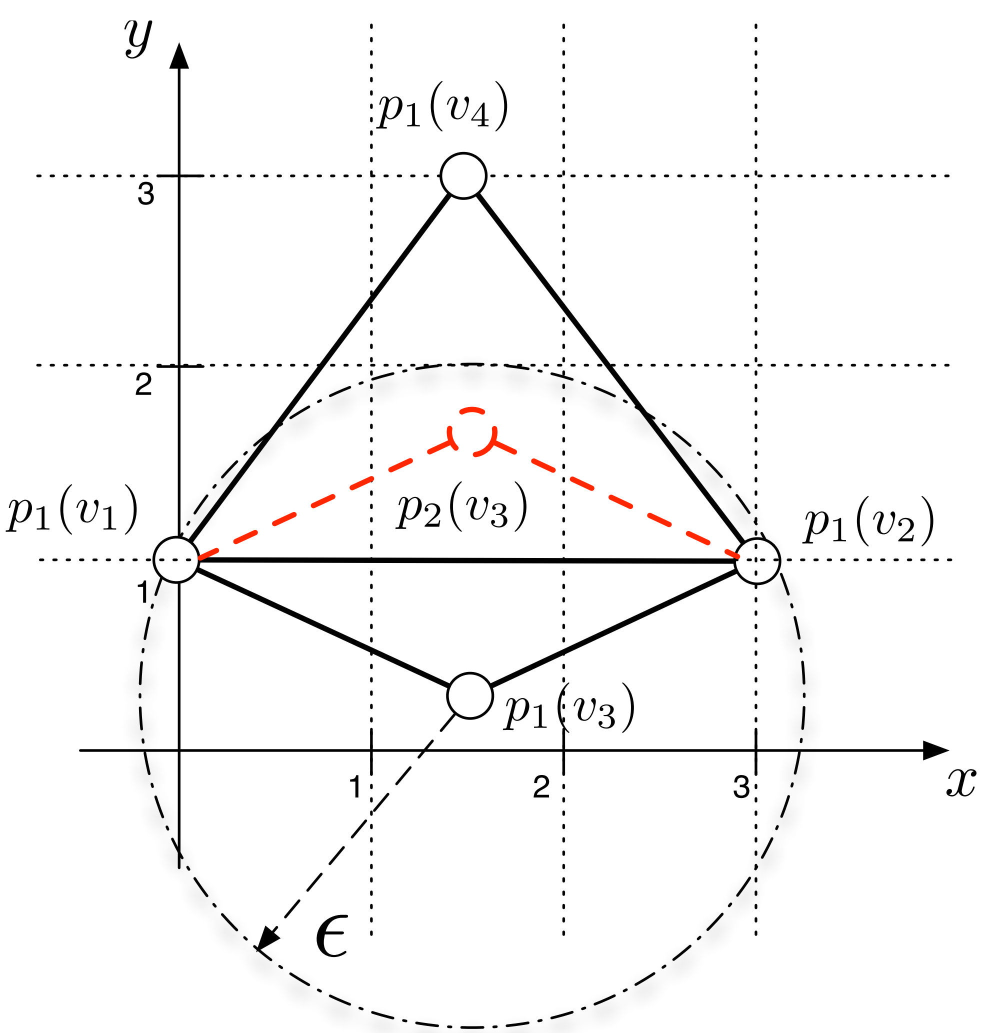
- translations
- rotations

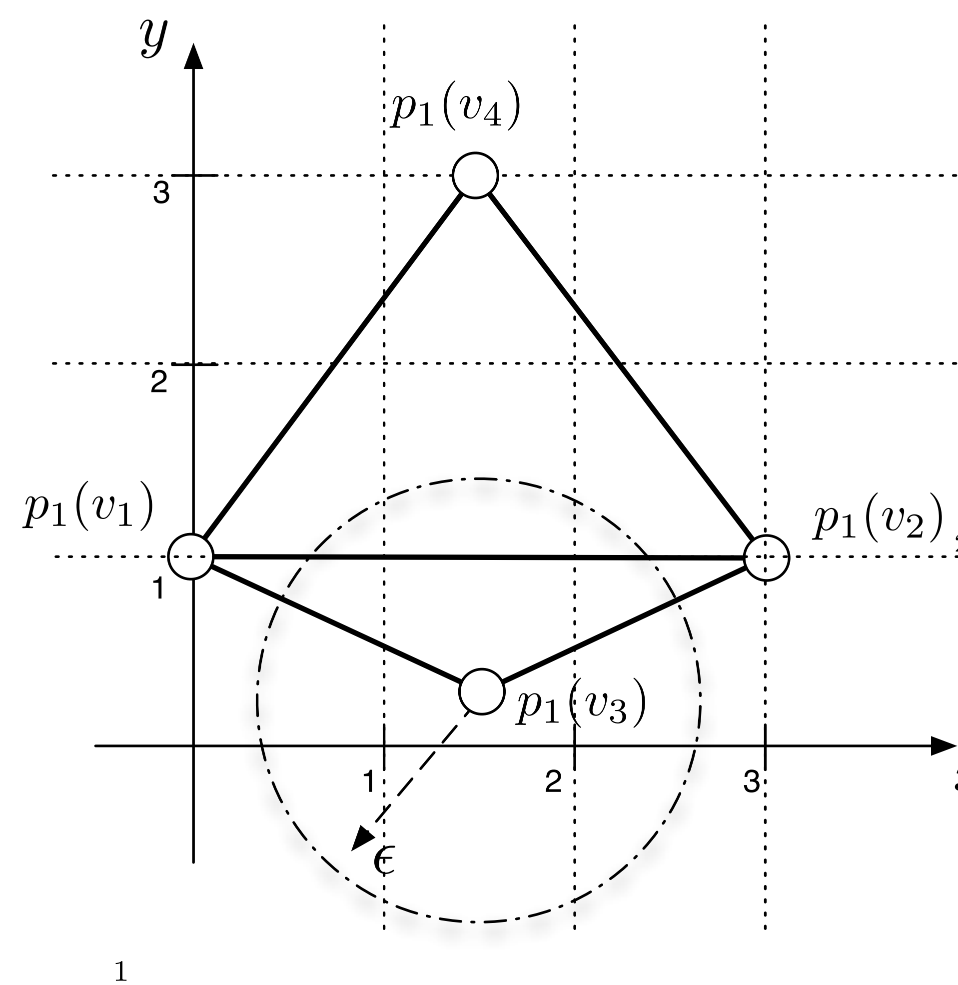


Bar-and-Joint Frameworks

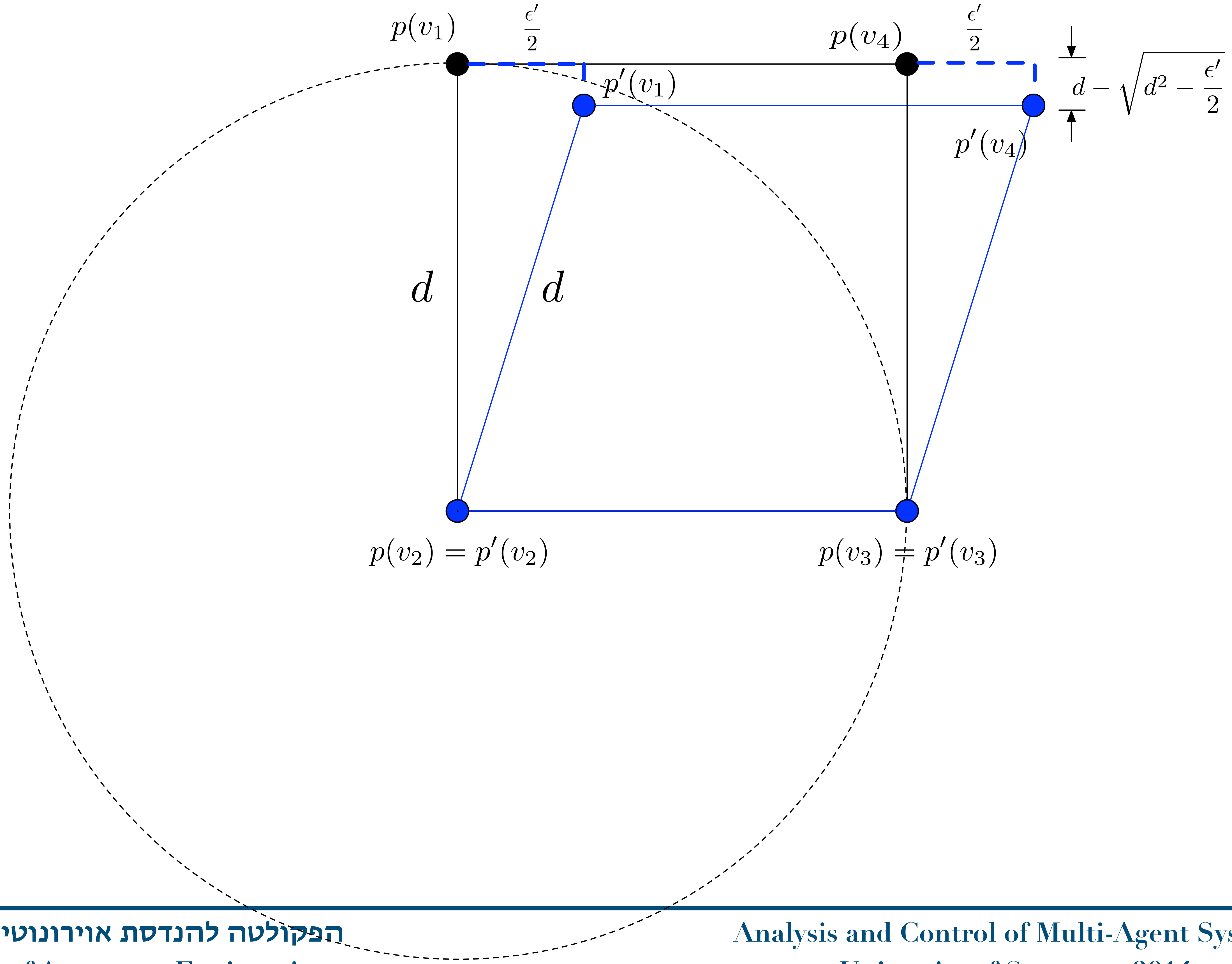
Definition

A framework $(\mathcal{G}, p_0)$ is *rigid* if there exists an $\epsilon > 0$ such that every framework $(\mathcal{G}, p_1)$ that is equivalent to $(\mathcal{G}, p_0)$ and satisfies $\|p_0(v) - p_1(v)\| < \epsilon$ for all $v \in \mathcal{V}$, is congruent to $(\mathcal{G}, p_0)$.





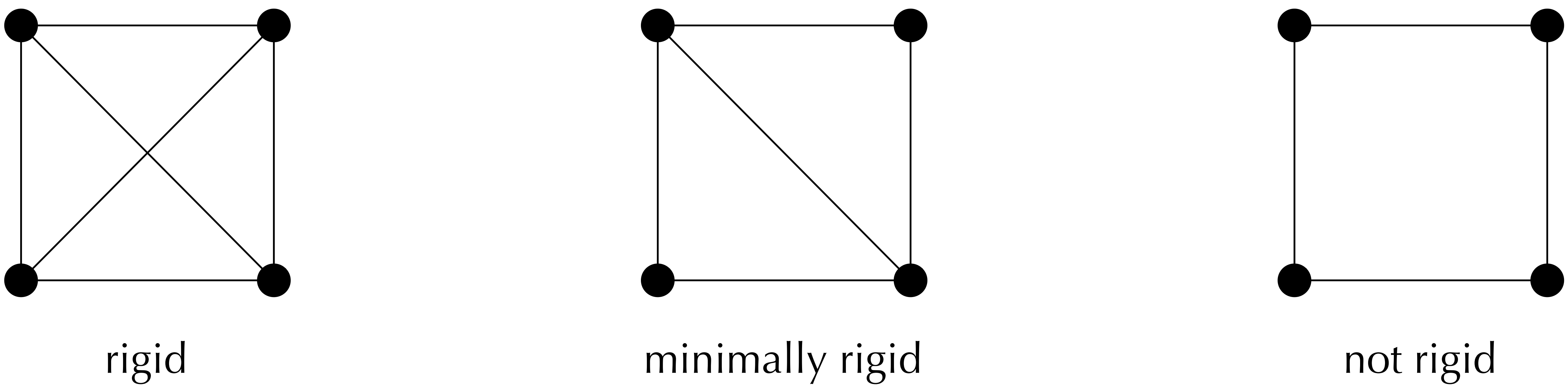
Bar-and-Joint Frameworks



Bar-and-Joint Frameworks

Definition

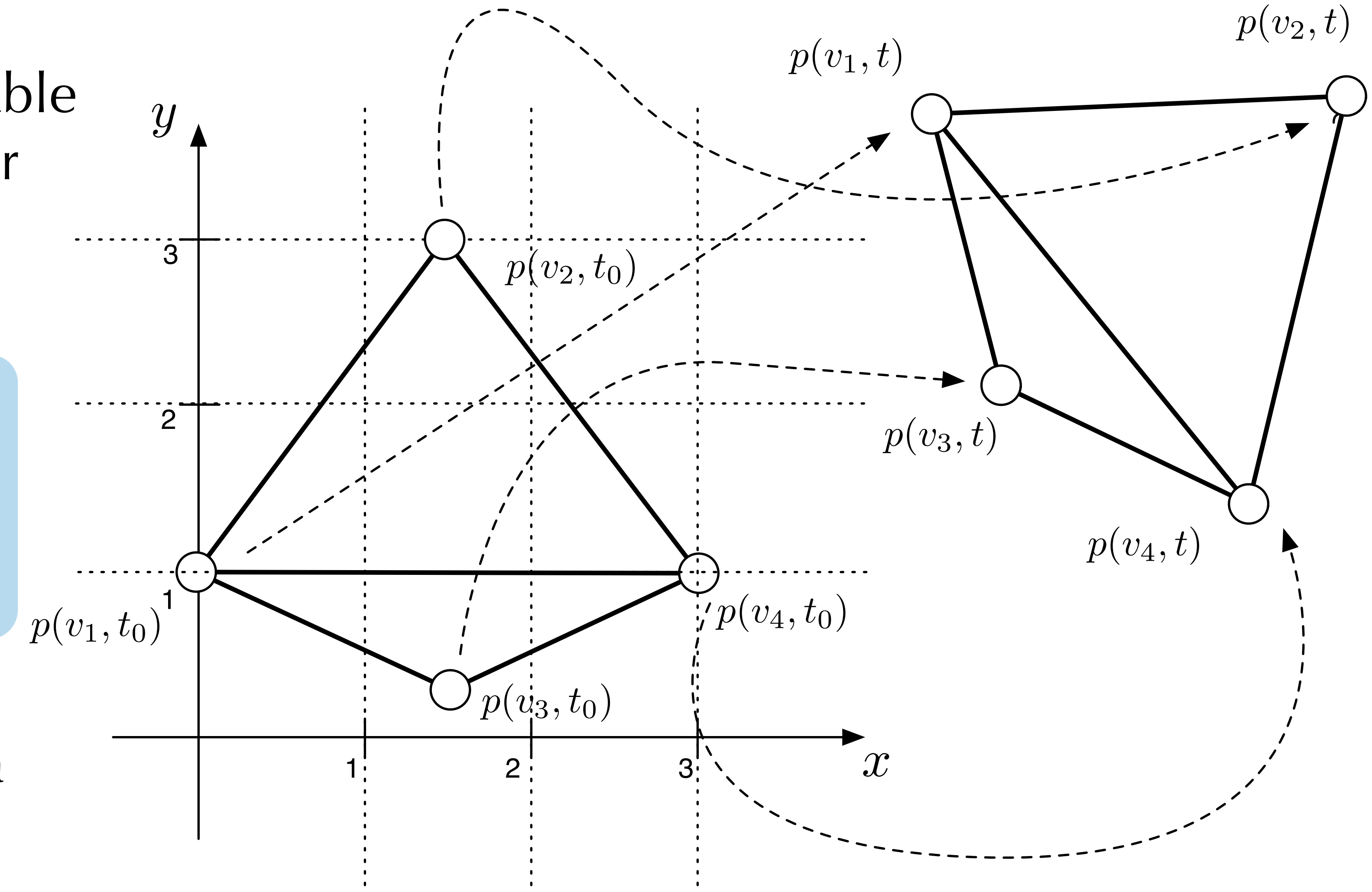
A *minimally rigid framework* is a rigid framework $(\mathcal{G}, p_0)$ such that the removal of any edge in $\mathcal{G}$ results in a non-rigid framework.



Bar-and-Joint Frameworks

parameterizing frameworks by a variable representing "time" allows to consider "motions" of a framework

$(\mathcal{G}, p, t)$



A trajectory is *edge consistent* if $\|p(v,t) - p(u,t)\|$ is constant for all $\{v,u\} \in \mathcal{E}$ and all t.

edge consistent trajectories generate a family of equivalent frameworks

$$\{p(u) \in \mathbb{R}^2 \mid \|p(u) - p(v)\|_2^2 = \ell_{uv}^2, \forall \{u,v\} \in \mathcal{E}\}$$

$$\Rightarrow \frac{d}{dt}\|x_u(t) - x_v(t)\| = 0, \forall \{u,v\} \in \mathcal{E}$$

$$\Rightarrow (\dot{x}_u(t) - \dot{x}_v(t))^T(x_u(t) - x_v(t)) = 0$$

infinitesimal motions

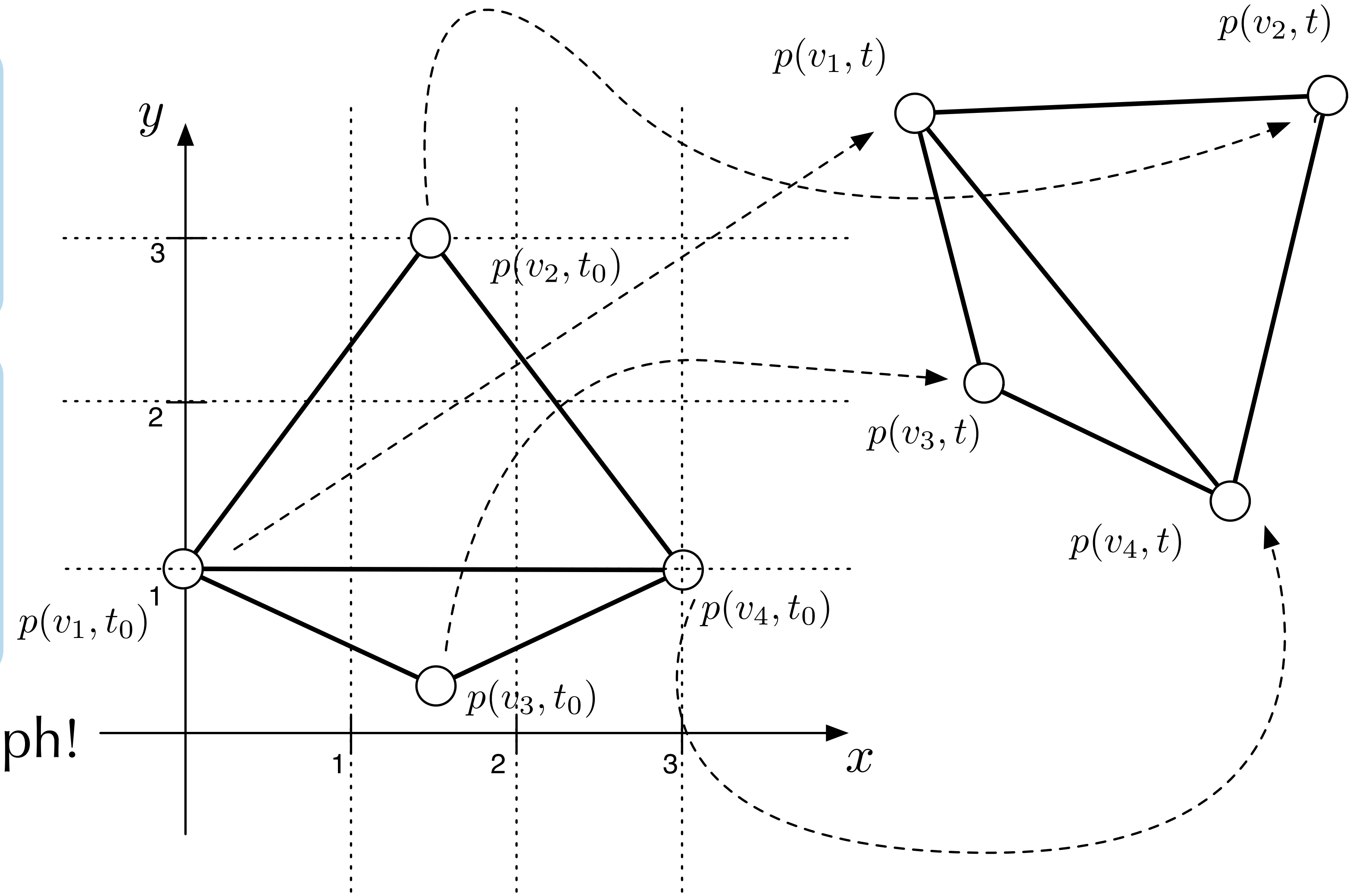
Bar-and-Joint Frameworks

Definition

A framework is ***infinitesimally rigid*** if every infinitesimal motion is *trivial*

Definition

A *graph* is ***generically rigid*** if it has an infinitesimally rigid framework realization



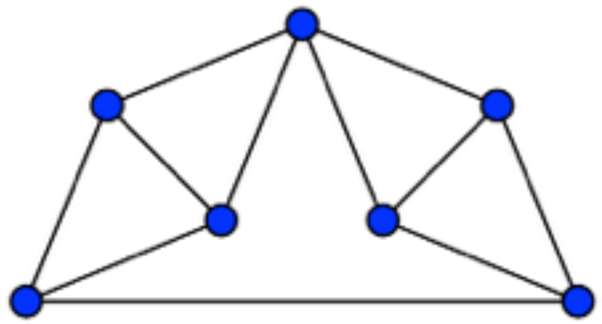
generic rigidity is a property of the graph!

How can we check if a graph is generically or infinitesimally rigid?

How can we construct generically rigid graphs?

Bar-and-Joint Frameworks

Definition. *A* Laman graph *is a graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with* $|\mathcal{E}| = 2|\mathcal{V}| - 3$ *such that all subgraphs of* k *vertices has at most* $2k - 3$ *edges.*



Theorem. *A graph is generically minimally rigid in* $\mathbb{R}^2$ *if and only if it is a Laman graph.*

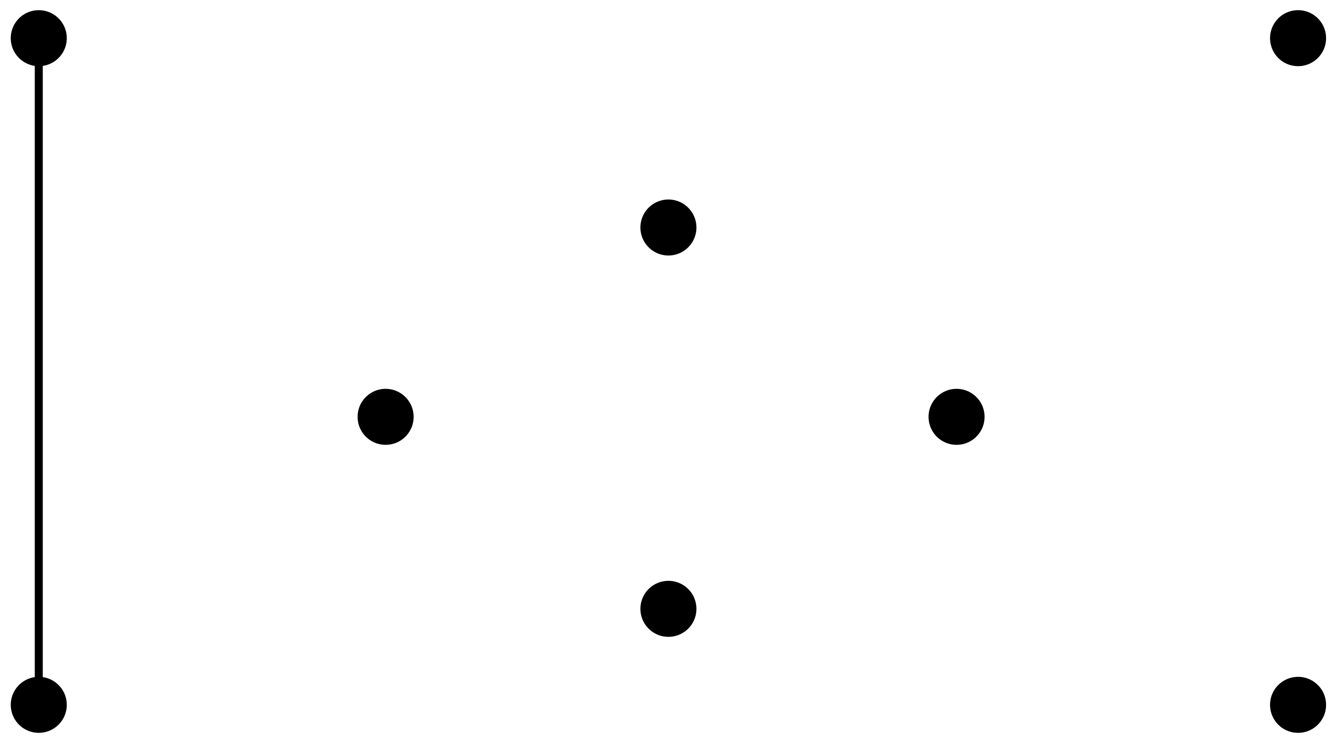
a *combinatorial* property! (i.e., hard to check)

Henneberg Constructions (1911) - a constructive method for generating all minimally generically rigid graphs in the plane

- vertex addition
- edge spliting

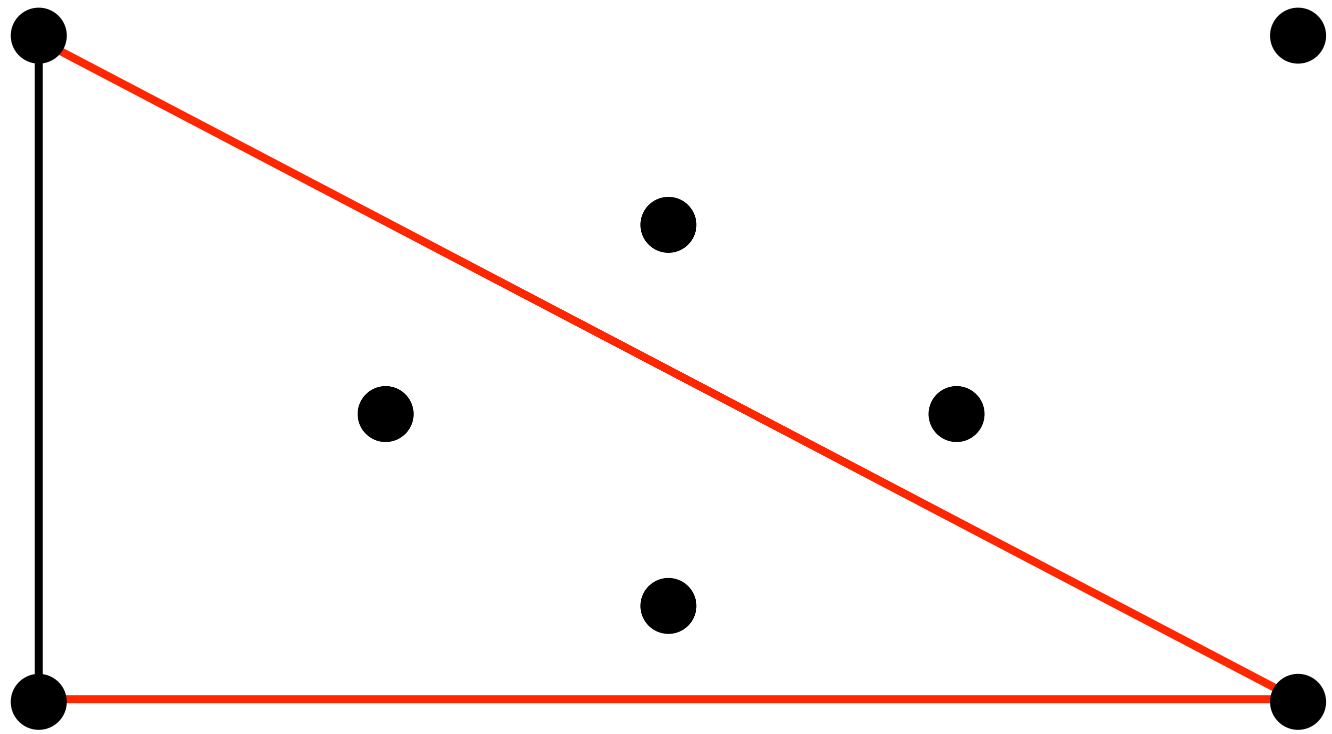
Henneberg Constructions

Example



Henneberg Constructions

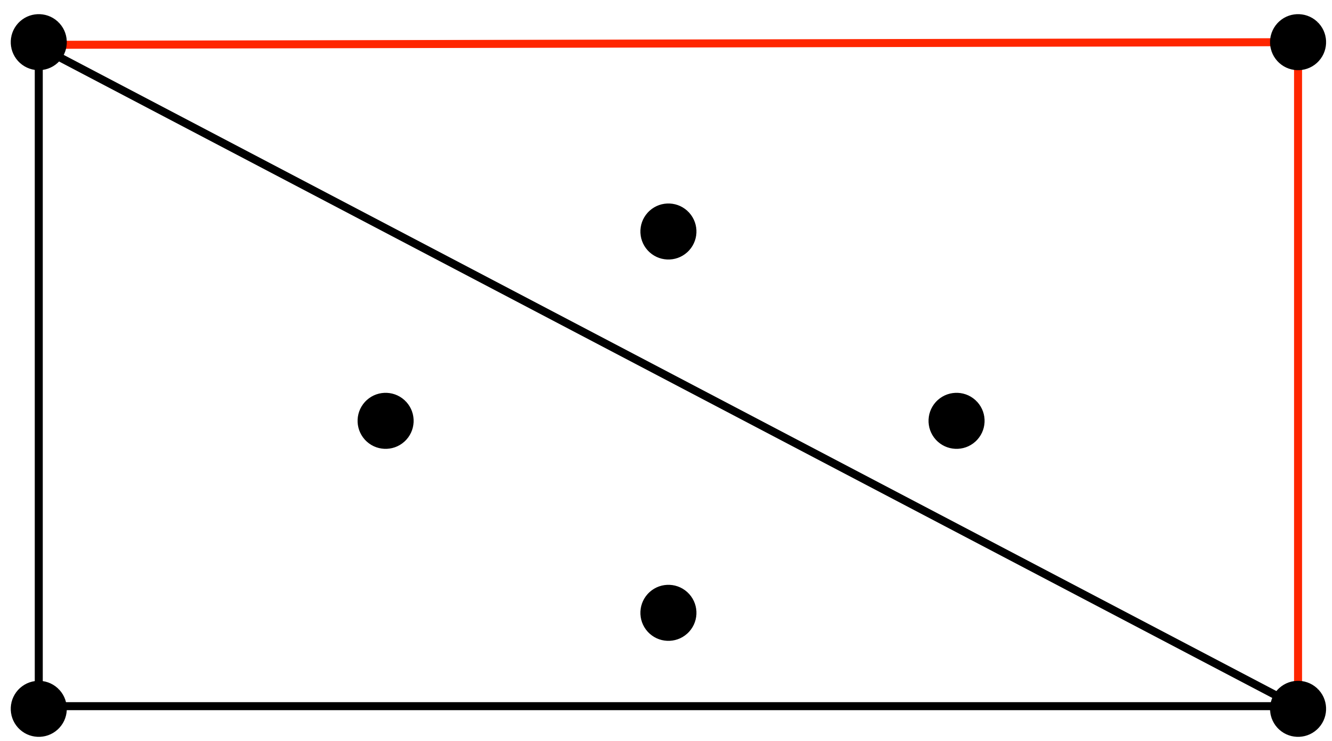
Example



Vertex Addition

Henneberg Constructions

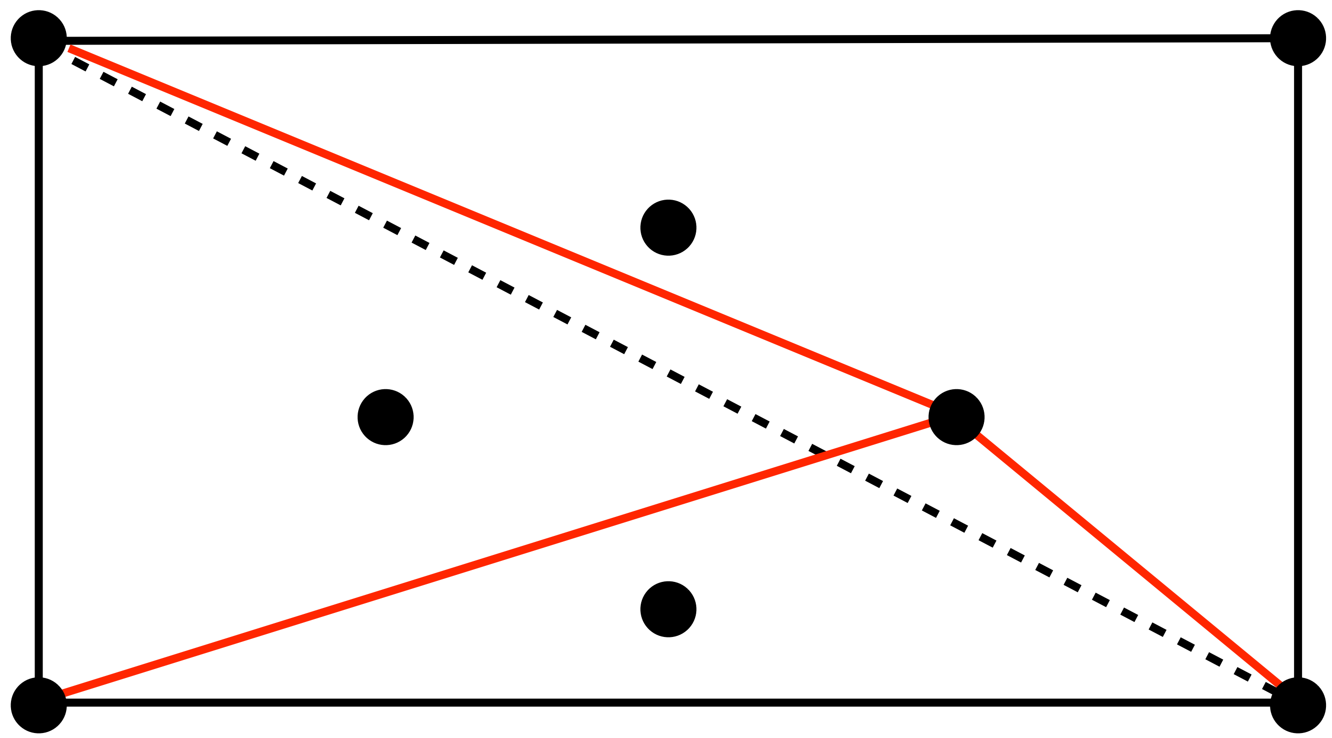
Example



Vertex Addition

Henneberg Constructions

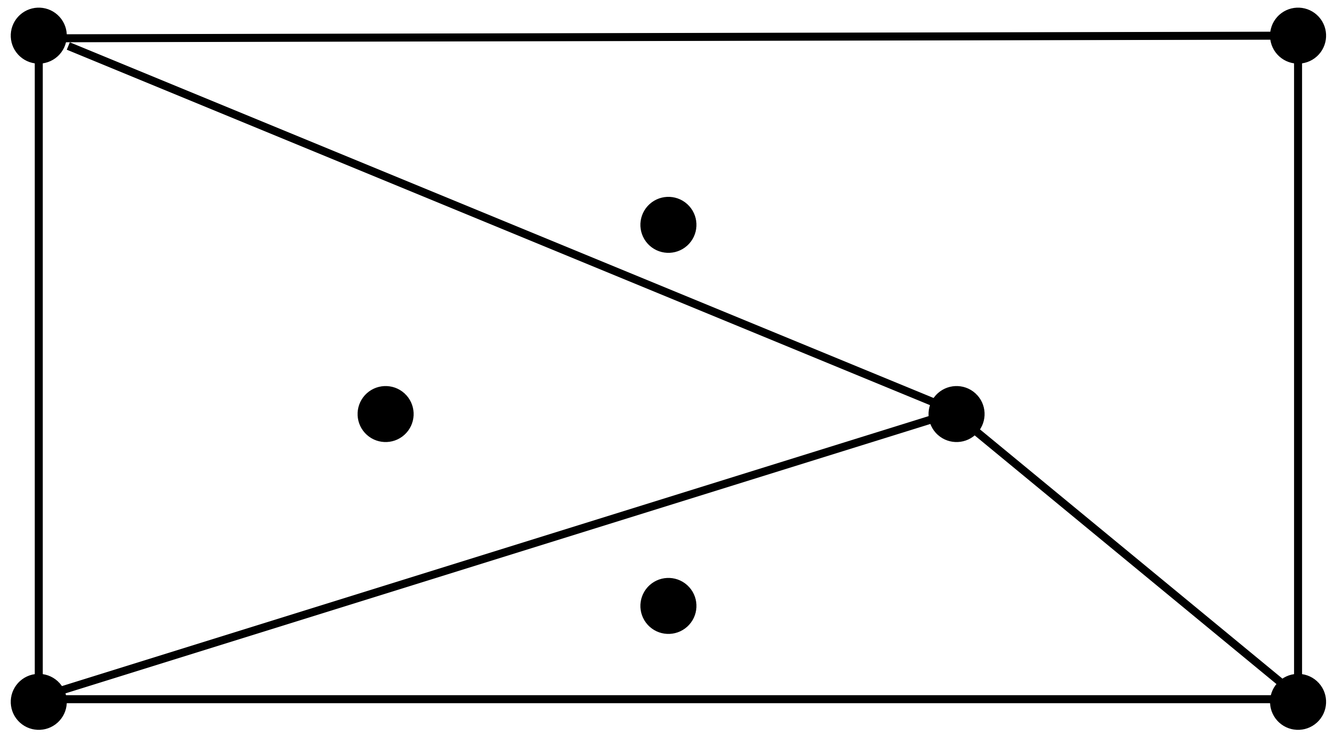
Example



Edge Splitting

Henneberg Constructions

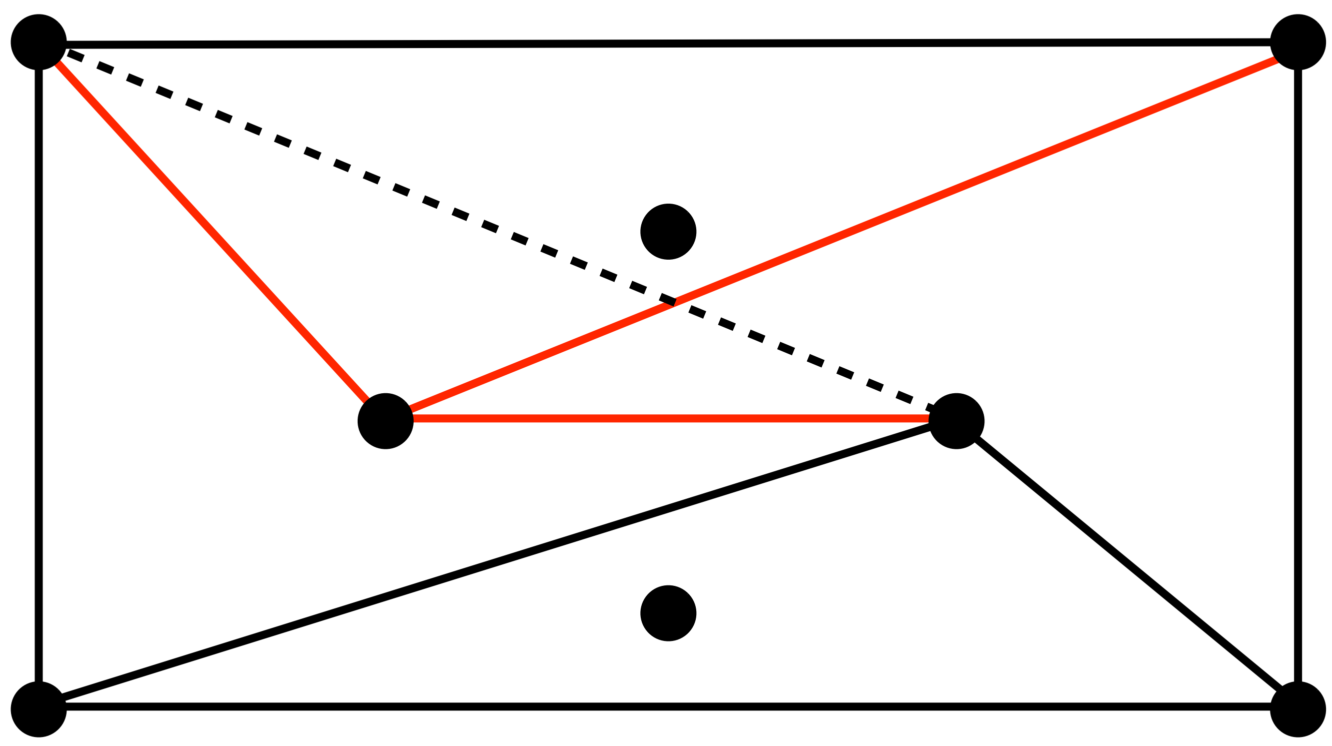
Example



Edge Splitting

Henneberg Constructions

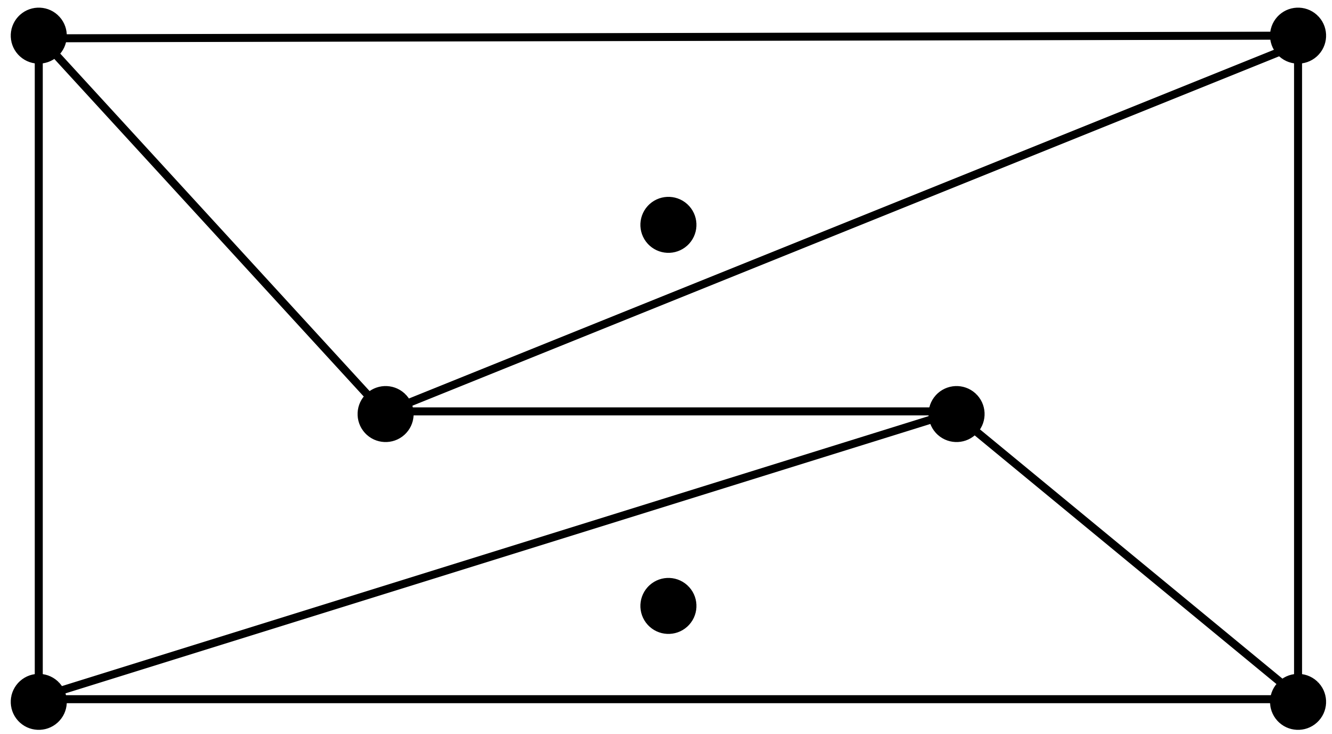
Example



Edge Splitting

Henneberg Constructions

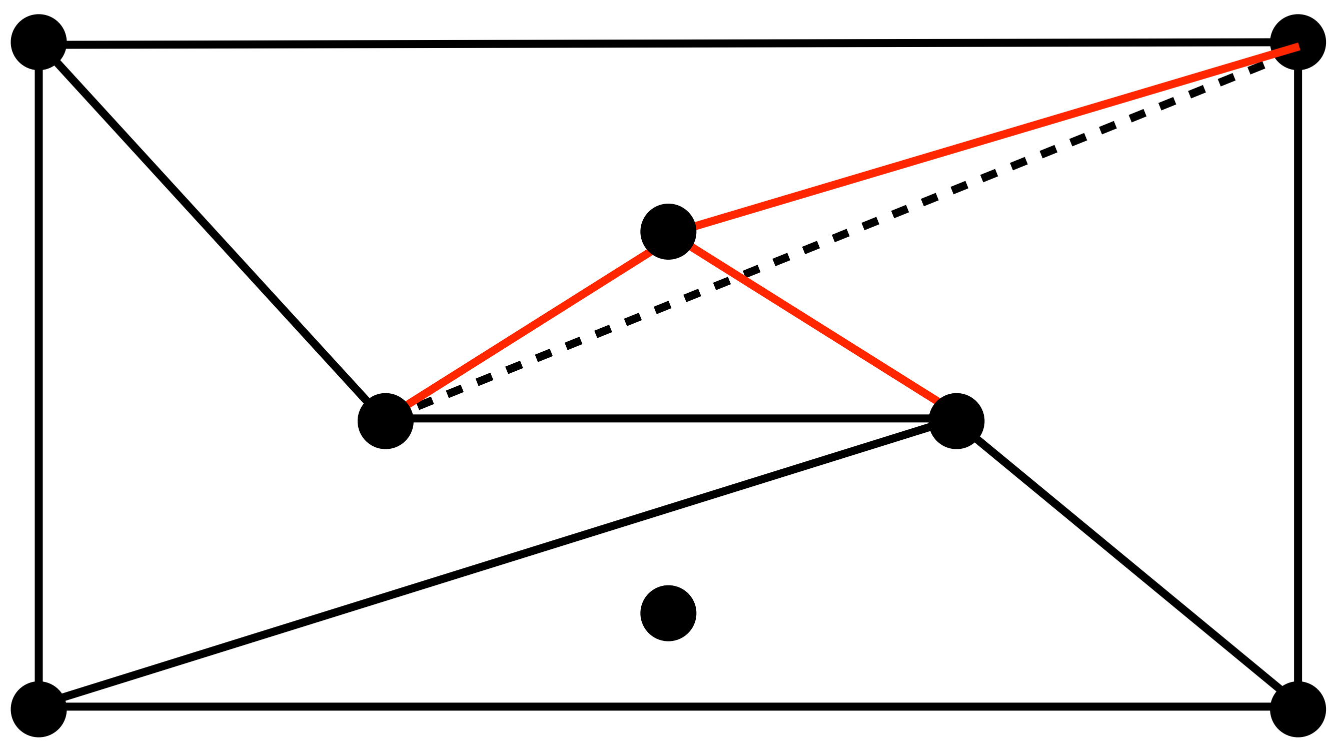
Example



Edge Splitting

Henneberg Constructions

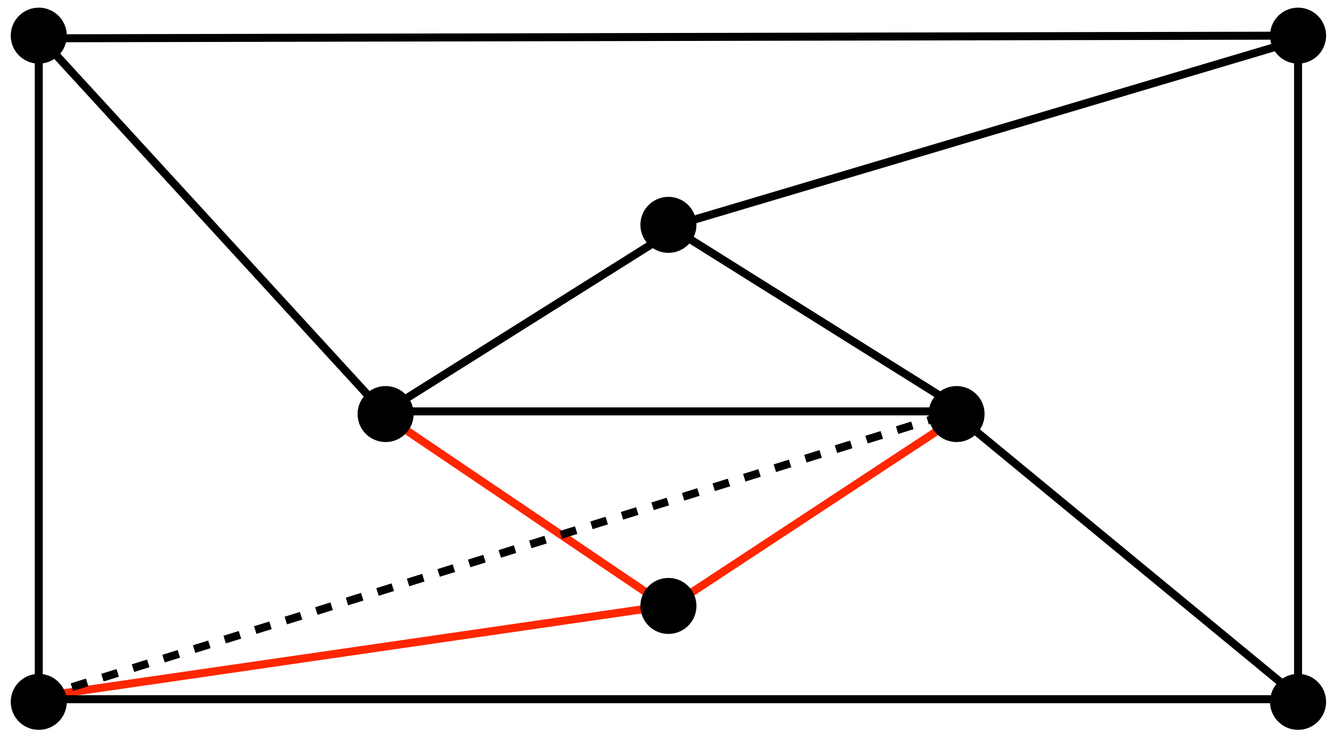
Example



Edge Splitting

Henneberg Constructions

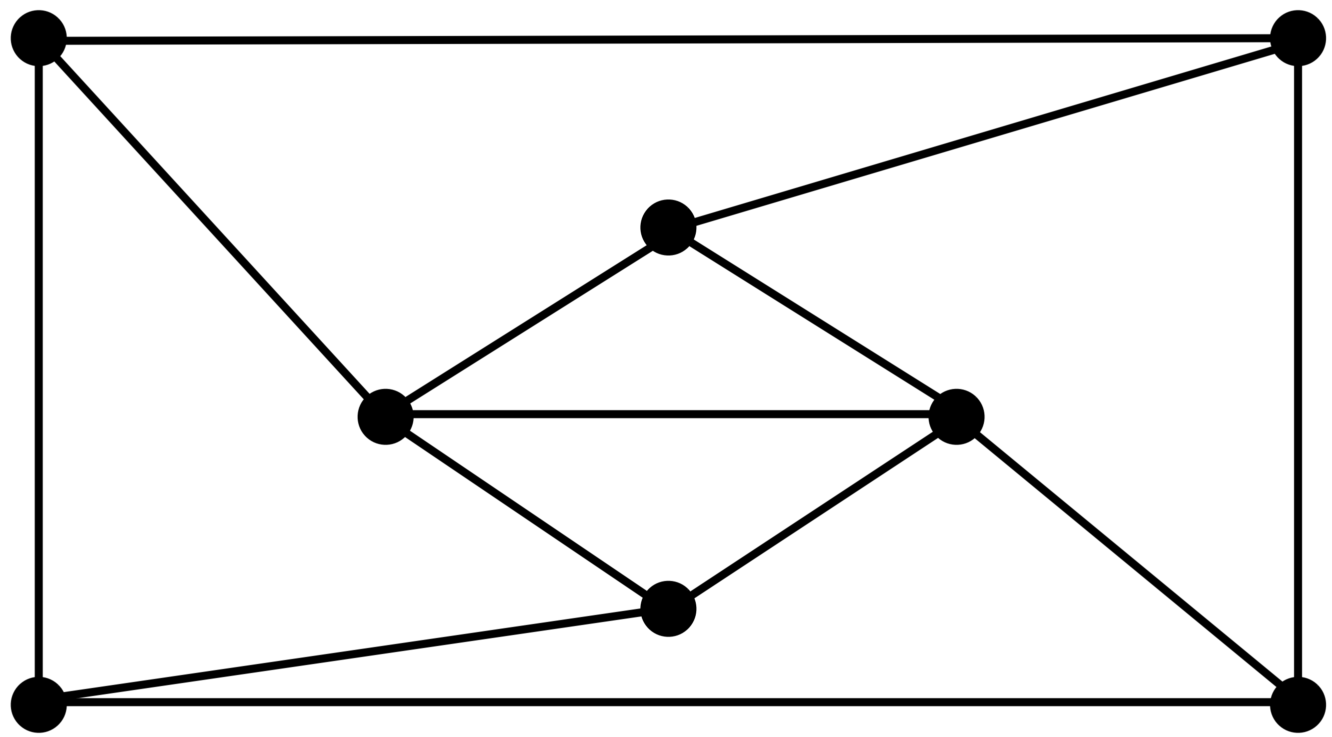
Example



Edge Splitting

Henneberg Constructions

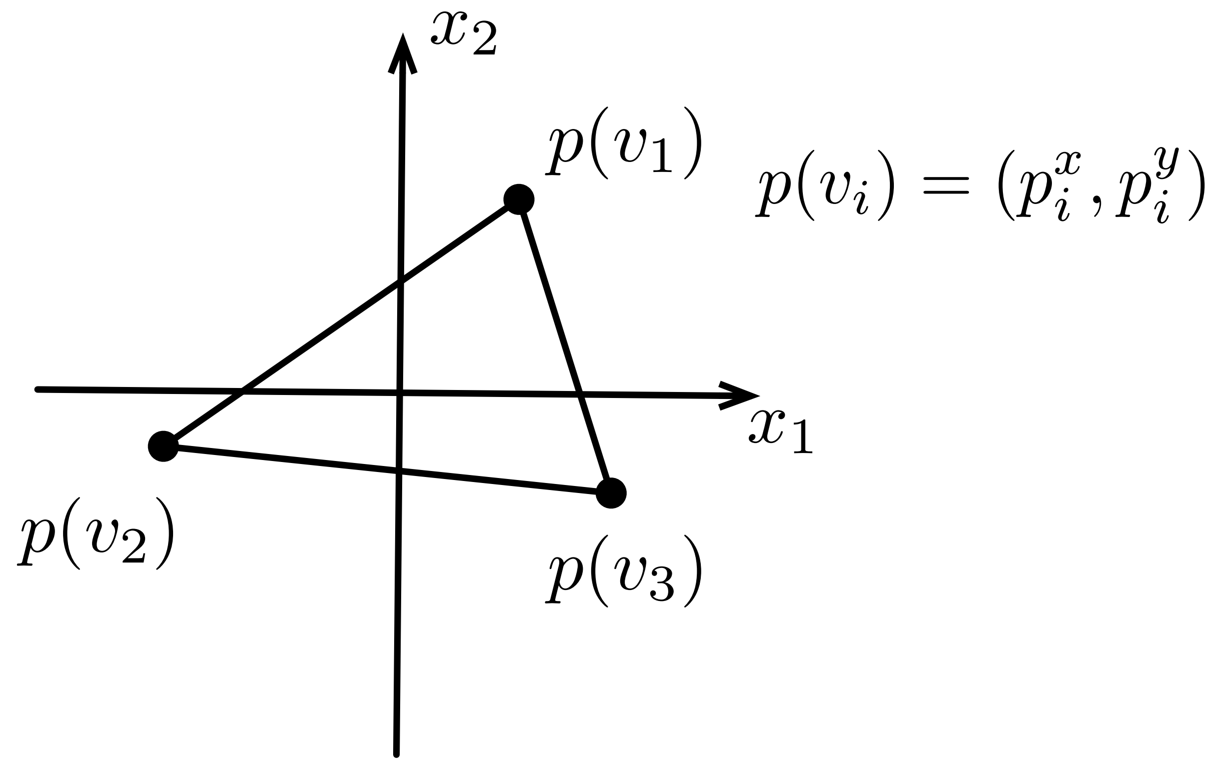
Example



The Rigidity Matrix

infinitesimal motions define a system of equations...

$$(\xi(v_i) - \xi(v_j))^T (p(v_i) - p(v_j)) = 0$$



The Rigidity Matrix

$$R(p) \in \mathbb{R}^{|\mathcal{E}| \times 2|\mathcal{V}|}$$

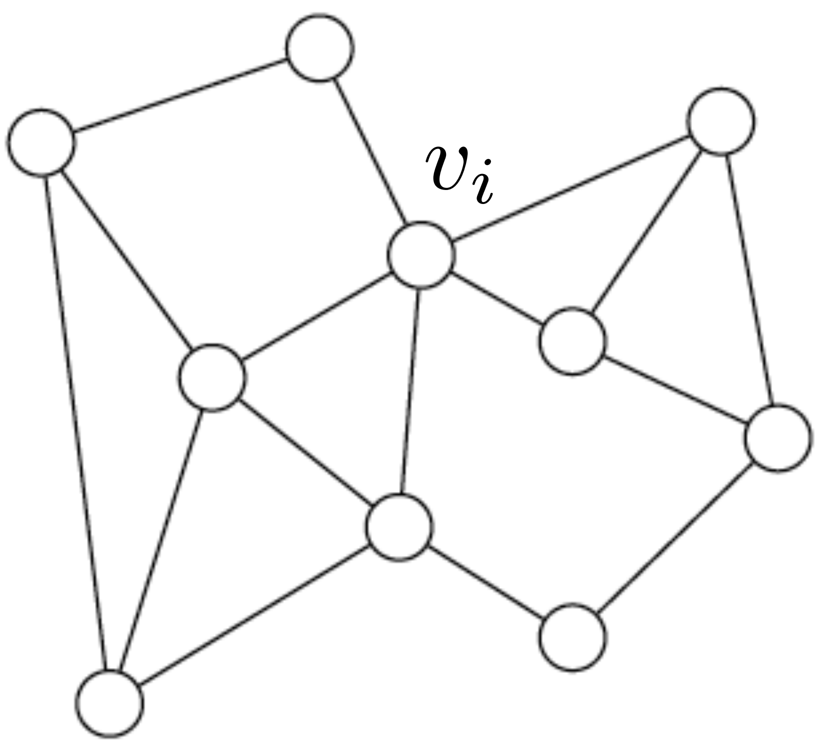
$$R(p) = \begin{bmatrix} p_1^x - p_2^x & p_1^y - p_2^y & p_2^x - p_1^x & p_2^y - p_1^y & 0 & 0 \\ p_1^x - p_3^x & p_1^y - p_3^y & 0 & 0 & p_3^x - p_1^x & p_3^y - p_1^y \\ 0 & 0 & p_2^x - p_3^x & p_2^y - p_3^y & p_3^x - p_2^x & p_3^y - p_2^y \end{bmatrix}$$

Lemma 1 (Tay1984) *A framework* $(\mathcal{G}, p)$ *is infinitesimally rigid if and only if* $\mathbf{rk}[R] = 2|\mathcal{V}| - 3$

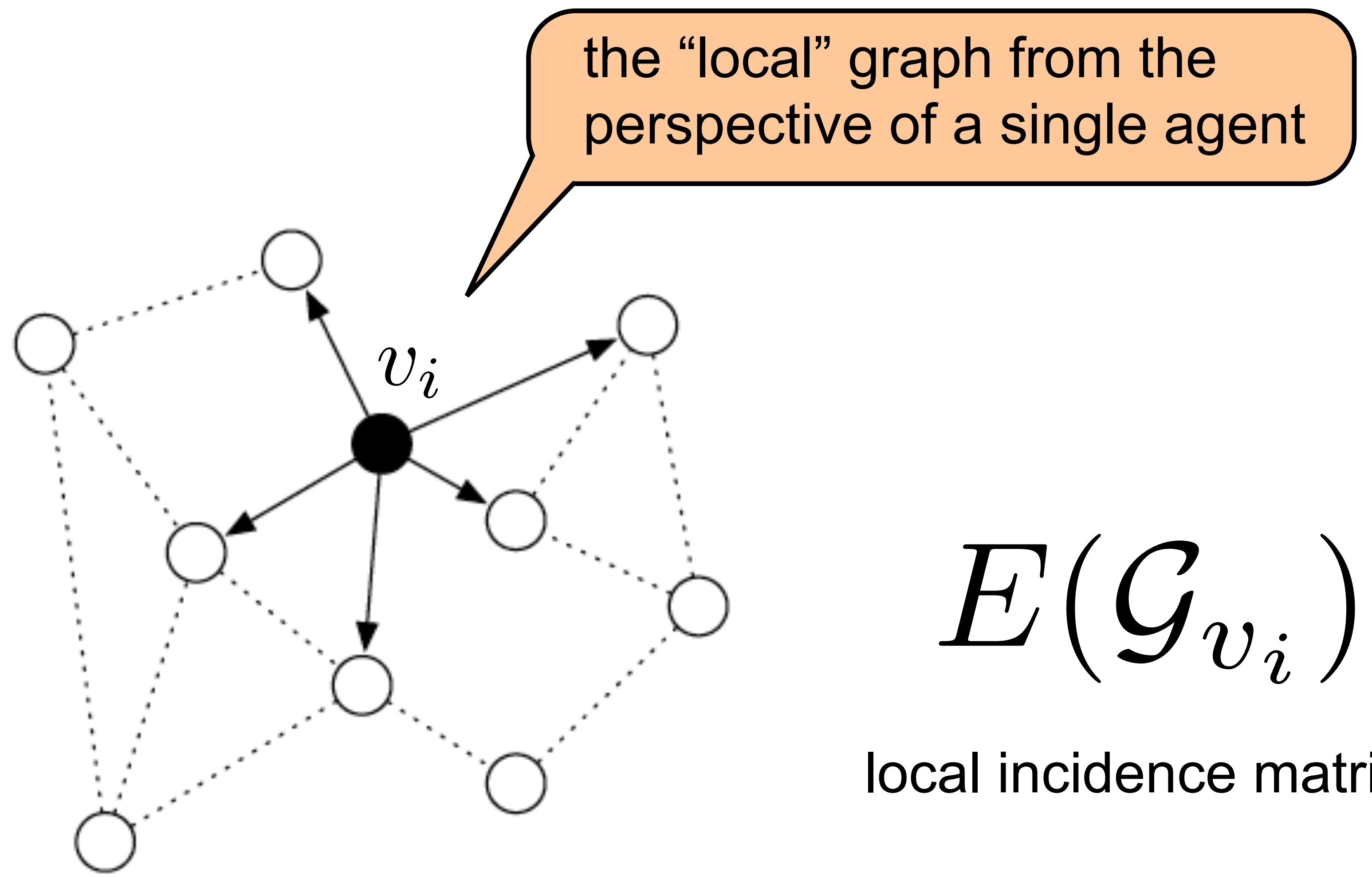
The Rigidity Matrix

The Rigidity Matrix

$$R(p) \in \mathbb{R}^{|\mathcal{E}| \times 2|\mathcal{V}|}$$



$\mathcal{G}$



$\mathcal{G}_{v_i}$

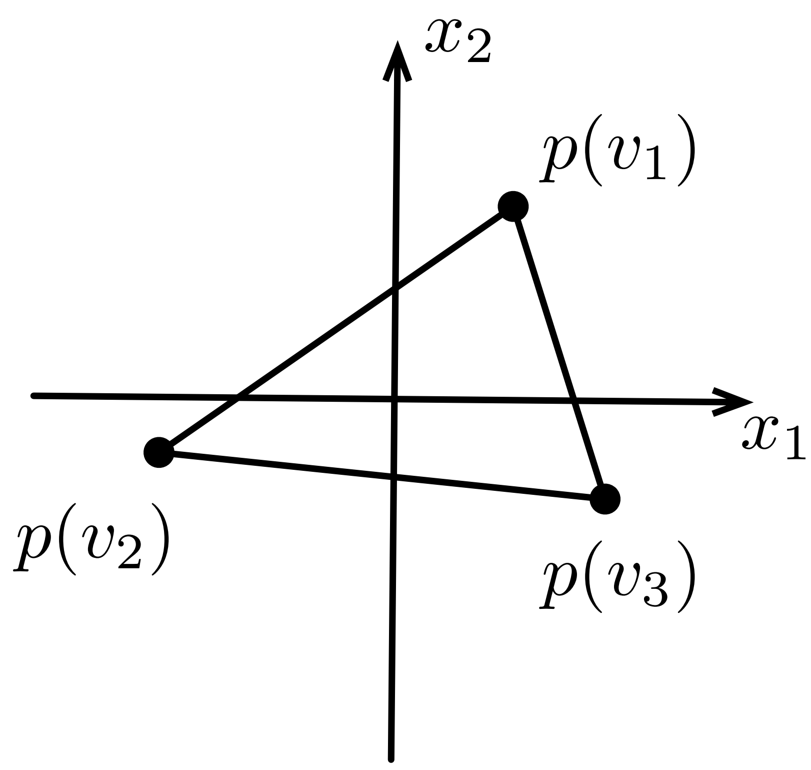
$$E(\mathcal{G}_{v_i})$$

local incidence matrix

The Rigidity Matrix

The Rigidity Matrix

$$R(p) \in \mathbb{R}^{|\mathcal{E}|\times 2|\mathcal{V}|}$$



$$p(v_i) = (p_i^x, p_i^y)$$

'local' incidence matrices

$$E(\mathcal{G}_1) = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad E(\mathcal{G}_2) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad E(\mathcal{G}_3) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R(p) = \begin{bmatrix} E(\mathcal{G}_1) & \cdots & E(\mathcal{G}_{|\mathcal{V}|}) \end{bmatrix} \left(I_{\mathcal{V}} \otimes p^{(x,y)} \right)$$

The Rigidity Matrix

The Symmetric Rigidity Matrix

$$\mathcal{R} = R(p)^T R(p)$$

λ_4 **the *Rigidity Eigenvalue***

a symmetric positive semi-definite matrix with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2|\mathcal{V}|}$$

Theorem *A framework is infinitesimally rigid if and only if the* rigidity eigenvalue *is strictly positive; i.e.,* $\lambda_4 > 0$.

proof: $$P\mathcal{R}P^T = (I_2 \otimes E(\mathcal{G})) \left[\begin{array}{cc} W_x & W_{xy} \\ W_{xy} & W_y \end{array} \right] (I_2 \otimes E(\mathcal{G})^T)$$

use properties of incidence matrix to show first three eigenvalues must be at the origin