Analysis and Control of Multi-Agent Systems

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Graph Rigidity and Formation Control

Graph Rigidity

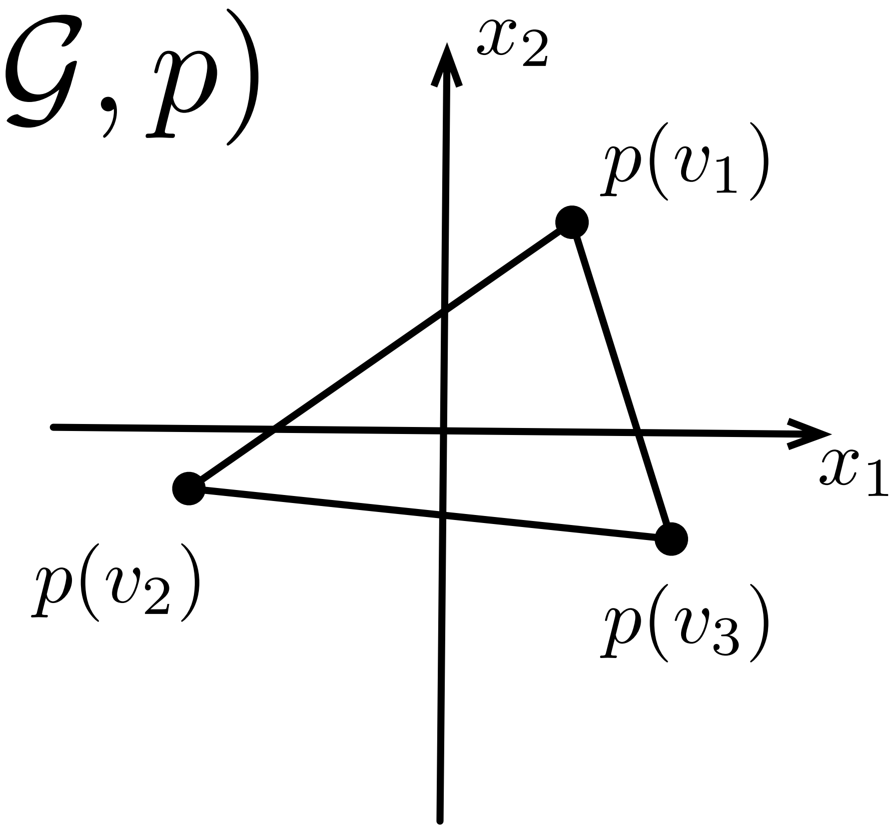
another approach…

Edge 'Distance' Function

$$f(p) = \frac{1}{2}\begin{bmatrix} \vdots \\ \|p(v_i) - p(v_j)\|^2 \\ \vdots \end{bmatrix} \in \mathbb{R}^{|\mathcal{E}|}$$

$\{v_i, v_j\} \in \mathcal{E}$

$(\mathcal{G}, p)$



the rigidity matrix is the 'linear' term in a Taylor series expansion of the edge function!

$$f(p+\delta_p) = f(p) + \frac{\partial f(p)}{\partial p}\delta_p + h.o.t.$$

The Rigidity Matrix

$$R(p) = \frac{\partial f(p)}{\partial p}$$

Graph Rigidity

Edge 'Distance' Function

$$f(p) = \frac{1}{2}\begin{bmatrix} \vdots \\ \|p(v_i) - p(v_j)\|^2 \\ \vdots \end{bmatrix} \in \mathbb{R}^{|\mathcal{E}|}$$

$$\{v_i, v_j\} \in \mathcal{E}$$

The Rigidity Matrix

$$R(p) = \frac{\partial f(p)}{\partial p}$$

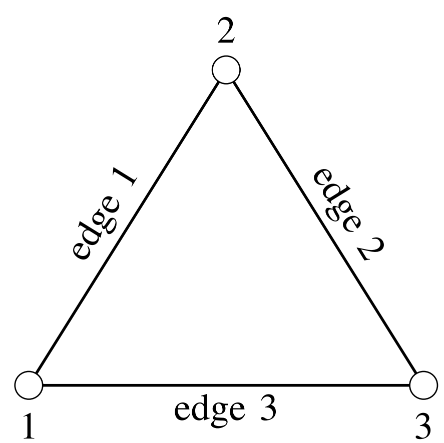
$$R(p) = \begin{bmatrix} E(\mathcal{G}_1) & \cdots & E(\mathcal{G}_{|\mathcal{V}|}) \end{bmatrix} \left(I_{\mathcal{V}} \otimes p^{(x,y)} \right) \quad \text{(last time)}$$

$$= \begin{bmatrix} \ddots & & \\ & \underbrace{p(v_i) - p(v_j)}_{e_{ij}} & \\ & & \ddots \end{bmatrix} \left(E(\mathcal{G})^T \otimes I \right)$$

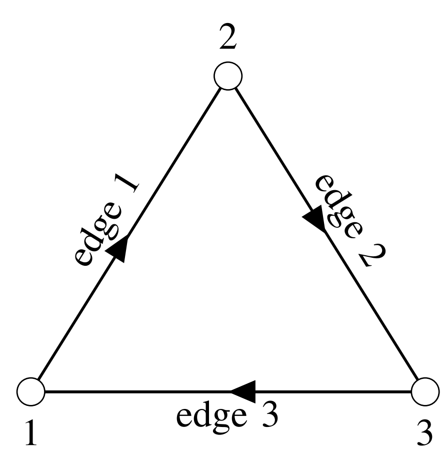
another form that separates the graph from the positions

Graph Rigidity

example...



(a) Undirected graph



(b) Oriented graph

$$R(p) = \begin{bmatrix} p_1^{\mathrm{T}} - p_2^{\mathrm{T}} & p_2^{\mathrm{T}} - p_1^{\mathrm{T}} & 0 \\ 0 & p_2^{\mathrm{T}} - p_3^{\mathrm{T}} & p_3^{\mathrm{T}} - p_2^{\mathrm{T}} \\ p_1^{\mathrm{T}} - p_3^{\mathrm{T}} & 0 & p_3^{\mathrm{T}} - p_1^{\mathrm{T}} \end{bmatrix}.$$

Denote $e_1 = p_2 - p_1$, $e_2 = p_3 - p_2$, and $e_3 = p_1 - p_3$.

$$R(p) = \begin{bmatrix} -e_1^{\mathrm{T}} & e_1^{\mathrm{T}} & 0 \\ 0 & -e_2^{\mathrm{T}} & e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} & 0 & -e_3^{\mathrm{T}} \end{bmatrix}$$

$$R(p) = \begin{bmatrix} e_1^{\mathrm{T}} & & \\ & \ddots & \\ & & e_m^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \otimes I_2$$

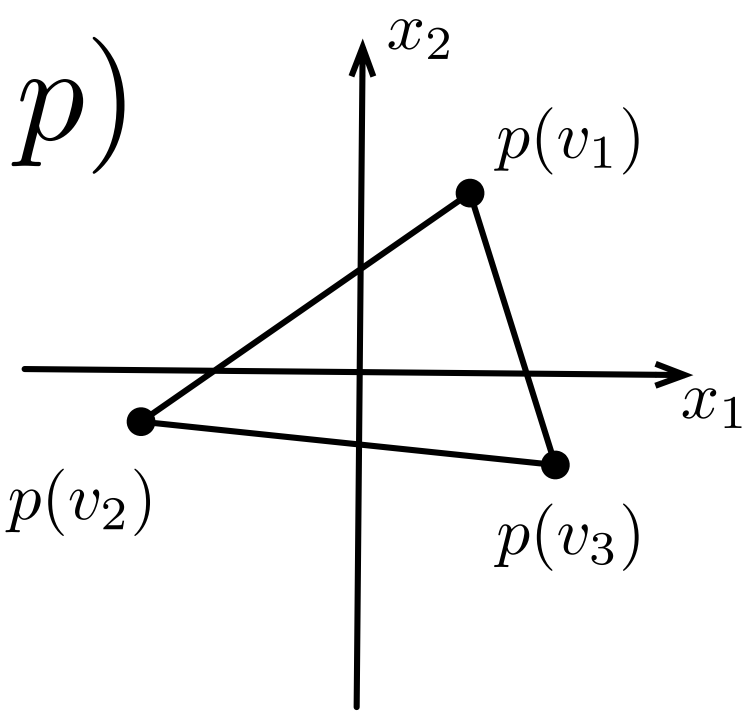
$$= \mathrm{diag}(e_i^{\mathrm{T}})(E^{\mathrm{T}} \otimes I_2)$$

The Rigidity Matrix

The Rigidity Matrix

$$R(p) = \frac{\partial f(p)}{\partial p}$$

$(\mathcal{G}, p)$



Lemma 1 (Tay1984) *A framework* $(\mathcal{G}, p)$ *is infinitesimally rigid if and only if* $\mathbf{rk}[R] = 2|\mathcal{V}| - 3$

A framework is *minimally infinitesimally rigid* (MIR) if it is infinitesimally rigid and minimally rigid.

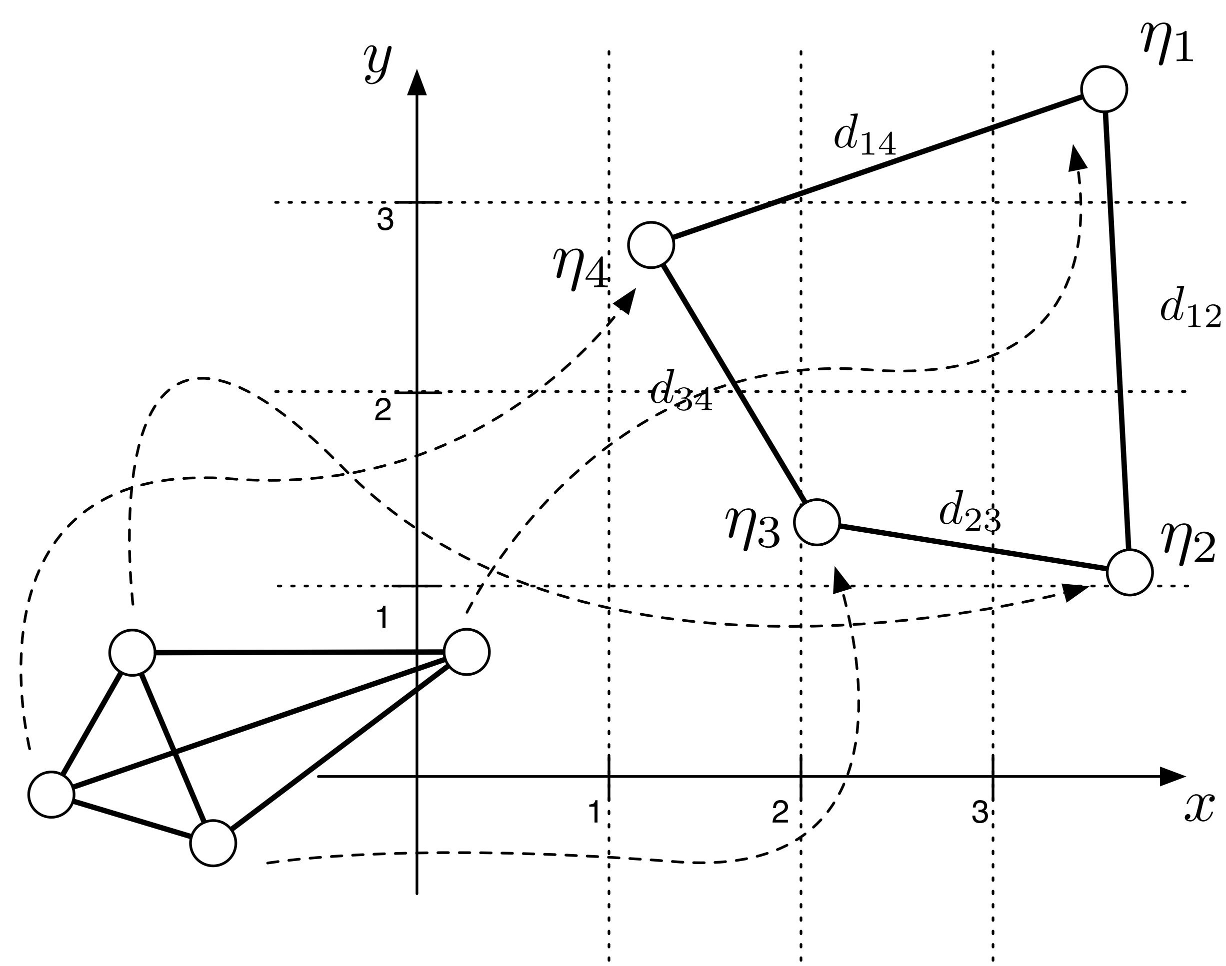
$$\Rightarrow \mathbf{rk}[R(p)] = 2|\mathcal{V}| - 3 = |\mathcal{E}|$$

MIR frameworks have ***full row rank***

Formation Control

A *formation* can be specified by inter agent distances

Rigidity is a way to ensure the formation is the desired "shape"



a collection of single-integrator agents

$$\dot{p}_i(t) = u_i(t)$$
$$p_i(t), u_i(t) \in \mathbb{R}^2$$

a sensing graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

a desired formation

$$\Phi = \{p \in \mathbb{R}^{2|\mathcal{V}|} \mid \|p_i - p_j\|^2 = d_{ij}^2, \forall \{i, j\} \in \mathcal{E}\}$$

Formation Control

a desired formation

$$\Phi = \{p \in \mathbb{R}^{2|\mathcal{V}|} \mid \|p_i - p_j\|^2 = d_{ij}^2, \forall \{i,j\} \in \mathcal{E}\}$$

design a distributed control such that

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\|^2 = d_{ij}^2$$

some notations…

$$e_{ij}(t) = e_k(t) = p_i(t) - p_j(t)$$

$$k = \{i,j\} \in \mathcal{E}$$

$$\lim_{t\to\infty} \|e_k\|^2 = d_k^2$$

$$\sigma_k = \|e_k\|^2 - d_k^2 = e_k^T e_k - d_k^2$$ distance error

Formation Control

Formation Potential

$$F(p) = \frac{1}{4}\sum_{k=1}^{|\mathcal{E}|}\left(\|e_k\|^2 - d_k^2\right)^2 = \frac{1}{4}\sum_{k=1}^{|\mathcal{E}|}\sigma_k^2$$
$$= \|f(p) - \frac{1}{2}d^2\|^2$$

A Gradient Dynamical System

$$\dot{p} = -\nabla F(p)$$

what does this system "look" like?
what are they equilibrium configurations? are they stable?
does this "solve" the formation control problem?

Formation Control

$$\dot{p} = -\nabla F(p)$$

$$= \frac{\partial F(p)}{\partial p}$$

$$= -\frac{1}{4}\sum_{k=1}^{|\mathcal{E}|} \frac{\partial \sigma_i^2}{\partial p}$$

$$= -\frac{1}{2}\sum_{k=1}^{|\mathcal{E}|} \frac{\partial \sigma_i}{\partial p}\sigma_i$$

$$= -R(p)^T \sigma_i = -R^T(p)R(p)p - R^T(p)d^2$$

symmetric rigidity matrix

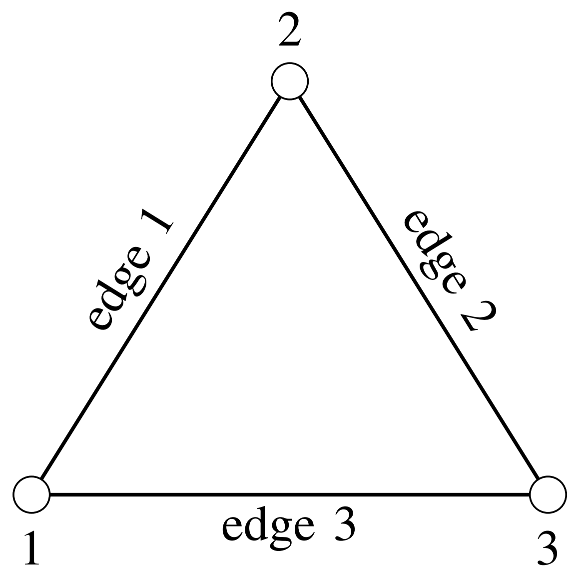
$$\dot{p}_i = -\sum_{j\sim i} \left(\|p_j - p_i\|^2 - d_{ij}^2\right)(p_i - p_j)$$

“looks” like a (state-dependent) weighted consensus protocol!

Formation Control

example... $\dot{p} = -R^{\mathrm{T}}(p)\sigma \iff \dot{p}_i = -\sum_{i=1}^{m} \sigma_{ij}(p_i - p_j)$

$$R(p) = \begin{bmatrix} p_1^{\mathrm{T}} - p_2^{\mathrm{T}} & p_2^{\mathrm{T}} - p_1^{\mathrm{T}} & 0 \\ 0 & p_2^{\mathrm{T}} - p_3^{\mathrm{T}} & p_3^{\mathrm{T}} - p_2^{\mathrm{T}} \\ p_1^{\mathrm{T}} - p_3^{\mathrm{T}} & 0 & p_3^{\mathrm{T}} - p_1^{\mathrm{T}} \end{bmatrix}$$



$$R^{\mathrm{T}}(p) = \begin{bmatrix} p_1 - p_2 & 0 & p_1 - p_3 \\ p_2 - p_1 & p_2 - p_3 & 0 \\ 0 & p_3 - p_2 & p_3 - p_1 \end{bmatrix}$$

$$R^{\mathrm{T}}(p)\sigma = \begin{bmatrix} \sigma_1(p_1 - p_2) + \sigma_3(p_1 - p_3) \\ \sigma_1(p_2 - p_1) + \sigma_2(p_2 - p_3) \\ \sigma_2(p_3 - p_2) + \sigma_3(p_3 - p_1) \end{bmatrix}$$

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

what are the equilibrium configurations?

$$0 = -R^T(p)R(p)p - R^T(p)d^2$$

1 $$0 = R(p)p - d^2 \Rightarrow (\|p_i - p_j\|^2 - d_{ij}^2) = 0$$

exactly the equilibrium we want!

2 $$0 = \begin{bmatrix} W_x(p) \\ W_y(p) \end{bmatrix} \left(\begin{bmatrix} W_x(p) & W_y(p) \end{bmatrix} (I_2 \otimes E^T)p - d^2 \right)$$

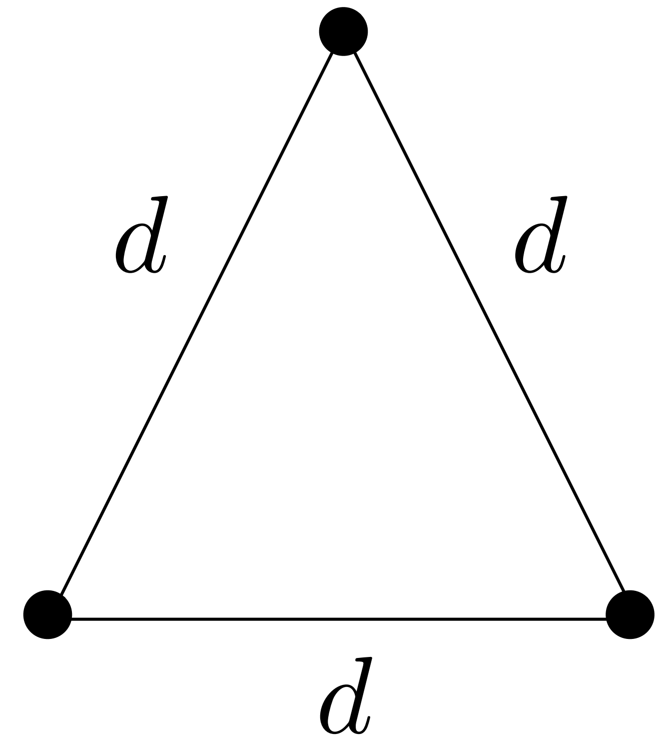
3 $$0 = (I_2 \otimes E) \left(\begin{bmatrix} W_x(p) \\ W_y(p) \end{bmatrix} \left(\begin{bmatrix} W_x(p) & W_y(p) \end{bmatrix} (I_2 \otimes E^T)p - d^2 \right) \right)$$

Formation Control - Stability Analysis

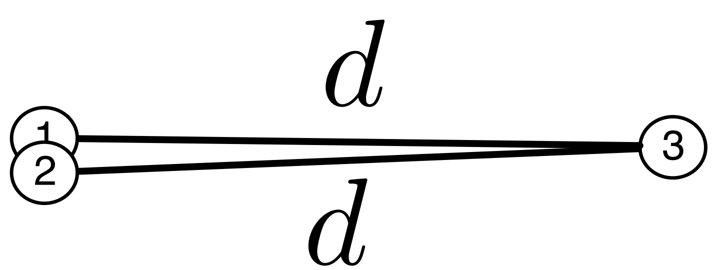
$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

example...

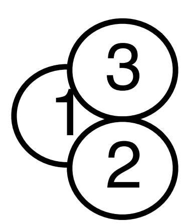
$$\begin{aligned}
\dot{p}_1 &= (\|p_1 - p_2\|^2 - d^2)(p_2 - p_1) + (\|p_1 - p_3\|^2 - d^2)(p_3 - p_1)\\
\dot{p}_2 &= (\|p_1 - p_2\|^2 - d^2)(p_1 - p_2) + (\|p_2 - p_3\|^2 - d^2)(p_3 - p_2)\\
\dot{p}_3 &= (\|p_1 - p_3\|^2 - d^2)(p_1 - p_3) + (\|p_2 - p_3\|^2 - d^2)(p_2 - p_3)
\end{aligned}$$



1. $\|p_i - p_j\|^2 = d^2$

2. 

'collinear'

3. 

'consensus'

the system has additional 'undesirable' equilibriums

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

example...

$$\begin{aligned}
\dot{p}_1 &= (\|p_1 - p_2\|^2 - d^2)(p_2 - p_1) + (\|p_1 - p_3\|^2 - d^2)(p_3 - p_1) \\
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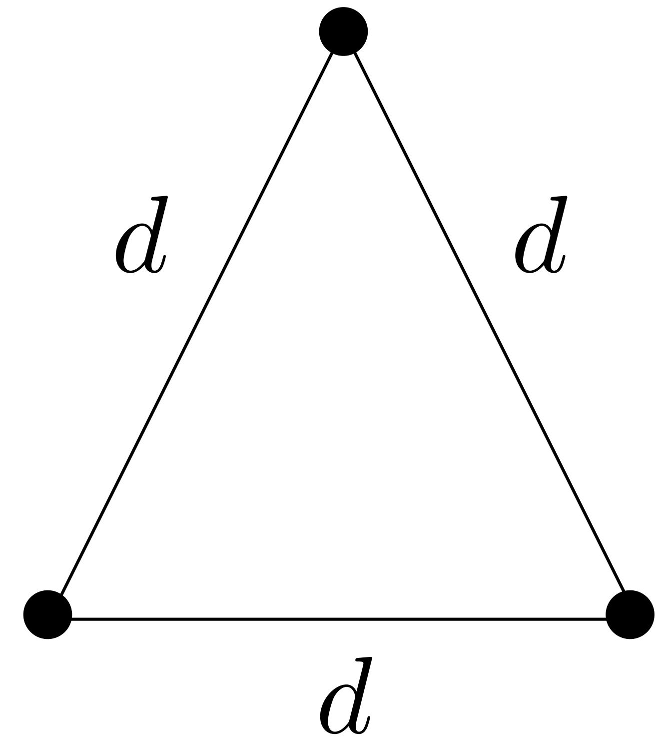
\end{aligned}$$



linearization about 'desired' equilibrium $\overline{p}$ $\quad (\|\overline{p}_i - \overline{p}_j\|^2 = d^2)$

$$\dot{\delta p}(t) = -\left(E(\mathcal{G}) \otimes I_2\right) \begin{bmatrix} (\overline{p}_1 - \overline{p}_2)(\overline{p}_1 - \overline{p}_2)^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\overline{p}_2 - \overline{p}_3)(\overline{p}_2 - \overline{p}_3)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\overline{p}_3 - \overline{p}_1)(\overline{p}_3 - \overline{p}_1)^T \end{bmatrix} \left(E(\mathcal{G})^T \otimes I_2\right) \delta p(t),$$

linearized state-matrix has 3 eigenvalues at 0 and remaining eigenvalues are real and negative

we can not conclude stability of equilibrium from linearized model!

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

a Lyapunov approach

$$F(p) = \frac{1}{4}\sigma^T\sigma$$

recall: the potential function defining a gradient dynamical system can serve as a Lyapunov function candidate!

$$\frac{d}{dt}F(p) = -\sigma^T R(p)R^T(p)\sigma \leq 0$$

negative semi-definite

$$\frac{d}{dt}F(p) = 0 \Leftrightarrow R^T(p)\sigma = 0$$

1. $\sigma = 0$
2. $\begin{bmatrix} \ddots & & \\ & p_i - p_j & \\ & & \ddots \end{bmatrix} \sigma \in \mathcal{N}[E \otimes I_2]$

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

a Lyapunov approach

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Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

Stability and full row rank

a rigidity matrix with full row rank (i.e. minimally infinitesimally rigid framework) $\Leftrightarrow \{\sigma \,|\, R^T(p)\sigma = 0\} = \{0\}$

$$\frac{d}{dt}F(p) = -\sigma^T \underbrace{R(p)R^T(p)}_{\text{a positive definite matrix!}}\sigma \leq 0 \quad \Rightarrow \frac{d}{dt}F(p) = 0 \Leftrightarrow \sigma = 0$$

Theorem

If the rigidity matrix has full row rank then the distributed distance-based formation control law (exponentially) converges to the specified formation set (locally).

Formation Control - Stability Analysis

Exponential stability…

$$\dot{x} = g(x, t)$$

if there exists a positive definite Lyapunov function satisfying $$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} \leq -kV(x)$$

then the nonlinear system is exponentially stable.

$$\dot{F}(p) = -\frac{\sigma^T R(p) R(p)^T \sigma}{F(p)} F(p)$$

$$= -\frac{\sigma^T R(p) R(p)^T \sigma}{\frac{1}{4}\sigma^T \sigma} F(p) \leq 4\lambda_{min}\left(R(p)R(p)^T\right) F(p)$$

rigidity eigenvalue!

Frameworks with Full Row Rank

A framework is *minimally infinitesimally rigid* (MIR) if it is infinitesimally rigid and minimally rigid.

$$\Rightarrow \mathbf{rk}[R(p)] = 2|\mathcal{V}| - 3 = |\mathcal{E}|$$

MIR frameworks have ***full row rank***

are there other (not infinitesimally rigid) frameworks that have full row rank?

what are the necessary and sufficient conditions needed to ensure the rigidity matrix of a framework has full row rank?

Frameworks with Full Row Rank

Definition

Given a framework $(\mathcal{G}, p)$, any set of scalars $w_{ij} = w_{ji}$ assigned to each edge of $\mathcal{G}$ is called a *stress* of the framework.

Definition

A stress $w = \begin{bmatrix} w_1 & \cdots & w_{|\mathcal{E}|} \end{bmatrix}^T$ is called a *self-stress* (or *equilibrium stress*) if

$$\sum_{j \sim i} w_{ij}(p_j - p_i) = 0, \ \forall i \in \mathcal{V}.$$

self-stresses mean the "forces" applied to a joint by neighboring joints (through the bars) are *balanced*

Frameworks with Full Row Rank

$$\sum_{j \sim i} w_{ij}(p_j - p_i) = 0,\ \forall i \in \mathcal{V} \Leftrightarrow w^T R(p) = 0$$

proof:

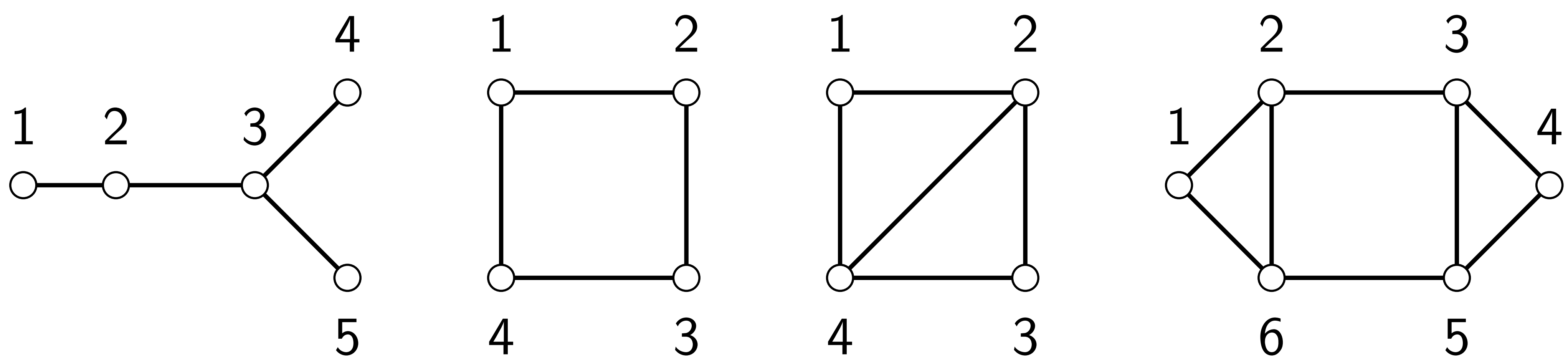
$$\sum_{j \in \mathcal{N}_i} \omega_{ij}(p_j - p_i) = 0, \forall i \in \mathcal{V}$$
$$\Leftrightarrow (E \otimes I_2)(W \otimes I_2)(E^{\mathrm{T}} \otimes I_2)p = 0$$
$$\Leftrightarrow (E \otimes I_2)(W \otimes I_2)e = 0$$
$$\Leftrightarrow (E \otimes I_2)\mathrm{diag}(e_i^{\mathrm{T}})\omega = 0$$
$$\Leftrightarrow R^{\mathrm{T}}(p)\omega = 0$$
$$\Leftrightarrow \omega^{\mathrm{T}} R(p) = 0$$

The space of self-stresses of a framework is the *left-null space* of the rigidity matrix!

Theorem

The rigidity matrix $R(p)$ of a framework $(\mathcal{G}, p)$ has full row rank if and only if $(\mathcal{G}, p)$ only supports zero self-stresses.

Frameworks with Full Row Rank



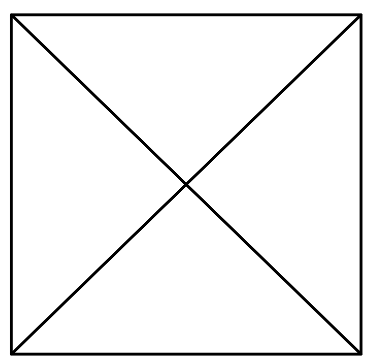
(a) Tree (b) Cycle (c) MIR (d) General

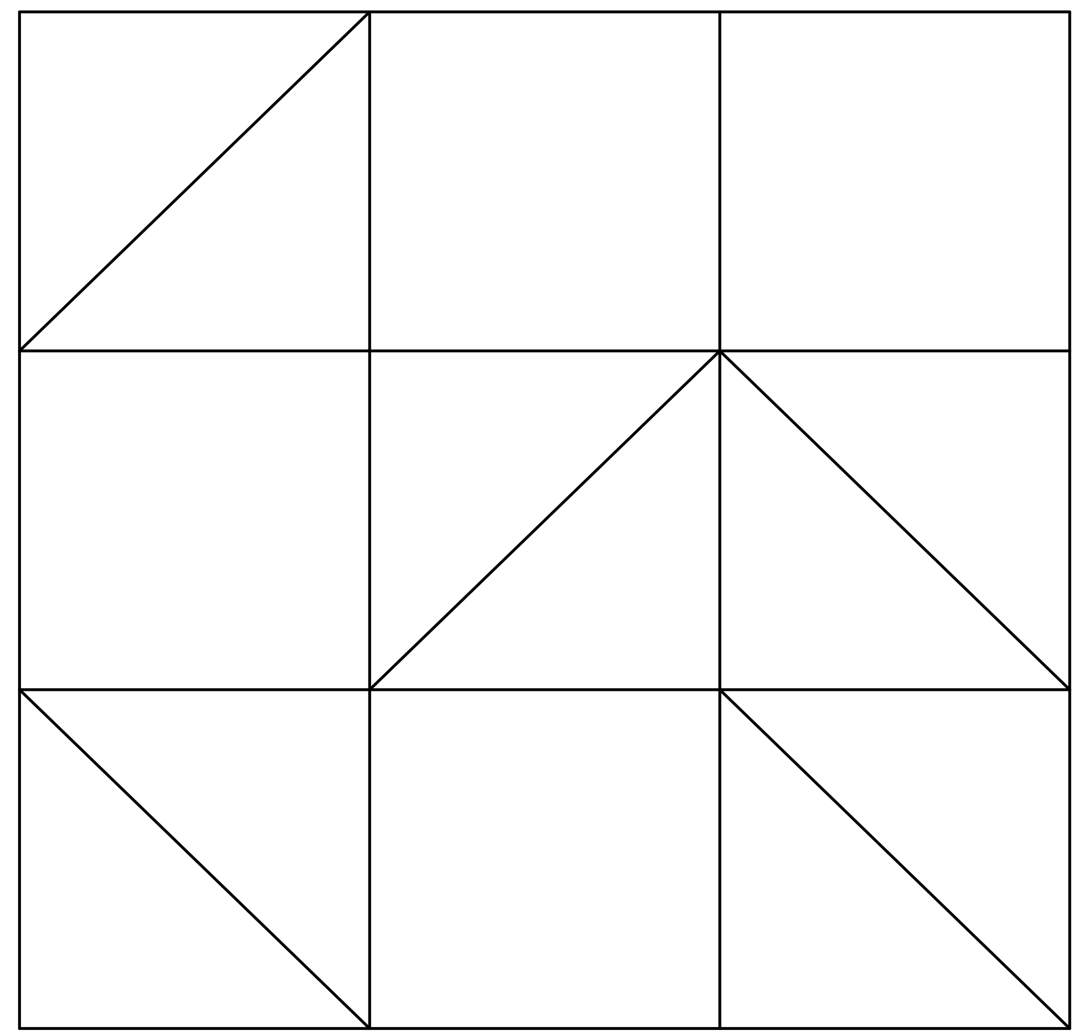
verify the following:

1. The rigidity matrix of a framework for an arbitrary spanning tree graph has full row rank
2. The rigidity matrix of a framework for a non-collinear cycle graph has full row rank
3. The rigidity matrix of a rigid framework has full row rank if and only if it is minimally infinitesimally rigid.

Frameworks with Full Row Rank

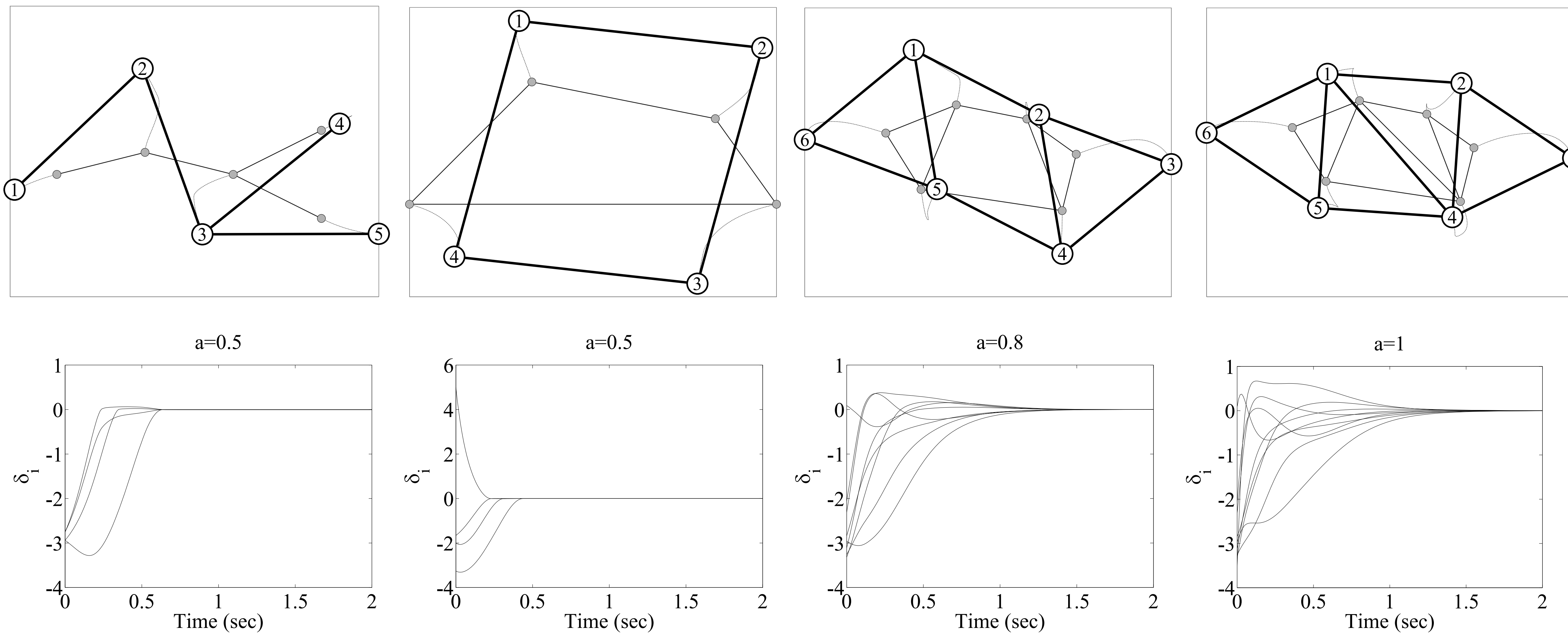
which framework has a rigidity matrix with full row rank?





Formation Control

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$



(a) Tree (b) Cycle (c) General flexible (d) MIR