Bearing-Constrained Formation Control using Bearing Measurements

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This paper studies distributed control of *bearing-constrained* multi-agent formations using *bearing-only* measurements. In order to analyze bearing-constrained formations, we first present a bearing rigidity theory that is applicable to arbitrary dimensions. Based on the proposed bearing rigidity theory, we investigate distributed bearing-only formation control in arbitrary dimensions in the presence of a global reference frame whose orientation is known to each agent. Using a Lyapunov approach, we show that the control law stabilizes infinitesimally bearing rigid formations almost globally and exponentially. The results are demonstrated with simulation examples.

I. Introduction

The precise distributed control of multi-agent formations is an enabling technology for a wide range of aerospace applications. For deep space and orbital applications, formation flying of spacecrafts can be used to perform hifidelity sensor fusion, such as in interferometry, with an increase in mission robustness and reduction of cost [1, 2]. Accurate formation control of unmanned aerial vehicles can lead to fuel savings and increased performance in problems related to cooperative tracking [3]. Despite the growing need for formation flying, there remain many challenges related to the design of distributed controllers for such systems. In particular, the sensing mediums available to a particular multi-agent mission will often dictate specialized control solutions. This has spurred interest across the aerospace and controls communities and is currently an active area of research [4–12].

One well-studied solution to the formation control problem employs the mathematical theory of *distance rigidity* [13–16]. Distance-based control schemes, however, are restrictive due to sensor costs and additional constraints such as the degradation of measurement fidelity as a function of range. A more cost-effective solution is to consider formation control using inter-agent *bearings*. In recent years, formation shape control with bearing constraints has also attracted much attention [7, 17–24]. The fundamental tool for analyzing bearing-constrained formations is the *bearing rigidity theory* (sometimes referred to as *parallel rigidity theory*) [18, 20, 21, 25, 26]. However, unlike distance rigidity which has been systematically studied since the 1970s [13–16], bearing rigidity theory has not been completely established yet. The existing studies on bearing rigidity [18,20,21,25] mainly focused on two-dimensional ambient spaces, whereas a systematic bearing rigidity theory that is applicable to arbitrary dimensions is still lacking. This motivates the subject of this work.

We consider in this paper a distributed bearing-only formation control problem where the formation is *bearing-constrained* and each agent has access to *bearing-only* measurements of their neighbors. The bearing measurements are directly applied in the formation

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control, and it is not required to estimate any other quantities from the bearing measurements. The research presented in this paper is applicable to a wide range of tasks such as vision-based cooperative control of autonomous vehicles [27–29] and satellite formation control using line-of-sight measurements [7].

Although bearing-only formation control has attracted much interest in recent years, many problems on this topic remain open. The studies in [19–21] considered bearing-constrained formation control in two-dimensional spaces, but required access to position or other measurements in the proposed control laws. The results reported in [28] only require bearing measurements. The bearing measurements, however, are used to estimate additional relative-state information such as distance ratios or scale-free coordinates. The works in [7, 17, 18, 22, 23] studied formation control with bearing measurements directly applied in the control. However, these results were applied to special formations, such as cyclic formations, and may not be extendable to arbitrary formation shapes. A very recent work reported in [24] solved bearing-only formation control for arbitrary underlying sensing graphs. This result, however, is valid only for two-dimensional formations. Bearing-only formation control in arbitrary dimensions with general underlying sensing graphs still remains an open problem.

The contributions of this paper are twofold. First, we propose a bearing rigidity theory that is applicable to arbitrary dimensions. Analogously to distance rigidity theory, we define notions of *bearing rigidity*, *global bearing rigidity*, and *infinitesimal bearing rigidity*. It is rigorously proved that bearing rigidity and global bearing rigidity are equivalent and infinitesimal bearing rigidity can uniquely determine the shape of a framework. Second, based on the proposed bearing rigidity theory, we investigate distributed bearing-only formation control in arbitrary dimensions in the presence of a global reference frame whose orientation is known to each agent (i.e., by endowing each agent with a compass). The case without global reference frames is addressed in [30]. Using a Lyapunov approach, we show that the proposed control law stabilizes infinitesimally bearing rigid formations almost globally and exponentially.

Notations Given $A_i \in \mathbb{R}^{p\times q}$ for $i = 1, \dots, n$, denote $\mathrm{diag}(A_i) \triangleq \mathrm{blkdiag}\{A_1, \dots, A_n\} \in \mathbb{R}^{np\times nq}$. Let $\mathrm{Null}(\cdot)$ and $\mathrm{Range}(\cdot)$ be the null space and range space of a matrix, respectively. Denote $I_d \in \mathbb{R}^{d\times d}$ as the identity matrix, and $\mathbf{1} \triangleq [1, \dots, 1]^{\mathrm{T}}$. Let $\|\cdot\|$ be the Euclidian norm of a vector or the spectral norm of a matrix, and $\otimes$ the Kronecker product.

An undirected graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with $m = |\mathcal{E}|$. The set of neighbors of vertex i is denoted as $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. An orientation of an undirected graph is the assignment of a direction to each edge. An oriented graph, denoted as $\mathcal{G}^\sigma = (\mathcal{E}^\sigma, \mathcal{V})$, is an undirected graph together with an orientation. If $(i, j) \in \mathcal{E}^\sigma$, vertex i is termed the tail and vertex j the head of the edge (i, j). The incidence matrix $H \in \mathbb{R}^{m\times n}$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows indexed by edges and columns by vertices: $[H]_{ki} = 1$ if vertex i is the head of edge k, $[H]_{ki} = -1$ if vertex i is the tail of edge k, and $[H]_{ki} = 0$ otherwise. For a connected graph, one always has $H\mathbf{1} = 0$ and $\mathrm{rank}(H) = n - 1$.

II. Bearing Rigidity in Arbitrary Dimensions

The existing literature on bearing rigidity theory is developed mainly for two-dimensional ambient spaces. In this section, we propose a bearing rigidity theory that is applicable to arbitrary dimensions. The connection between the bearing rigidity and the popular distance rigidity will also be explored.

The bearing rigidity theory is built on the notion of a *framework* consisting of an undirected graph, $\mathcal{G}$, and a *configuration*, p; we denote a framework as $\mathcal{G}(p)$. A *configuration* in $\mathbb{R}^d$ ($d \geq 2$) is a finite collection of n ($n \geq 2$) points, $p = [p_1^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$ with $p_i \neq p_j$ for all $i \neq j$. Each vertex in the graph corresponds to a point in the configuration. For a framework $\mathcal{G}(p)$, define the *edge vector* and the *bearing*, respectively, as

$$e_{ij} \triangleq p_j - p_i, \quad g_{ij} \triangleq e_{ij}/\|e_{ij}\|, \quad \forall (i,j) \in \mathcal{E}. \tag{1}$$

Note $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$. It is often helpful to consider an arbitrary oriented graph, $\mathcal{G}^\sigma = \{\mathcal{V}, \mathcal{E}^\sigma\}$, and express the edge vector and the bearing for the kth directed edge $(i,j) \in \mathcal{E}^\sigma$ as

$$e_k \triangleq p_j - p_i, \quad g_k \triangleq e_k/\|e_k\|, \quad \forall k \in \{1, \ldots, m\}. \tag{2}$$

Let $e = [e_1^{\mathrm{T}}, \ldots, e_m^{\mathrm{T}}]^{\mathrm{T}}$ and $g = [g_1^{\mathrm{T}}, \ldots, g_m^{\mathrm{T}}]^{\mathrm{T}}$. Note e satisfies $e = \bar{H}p$ where $\bar{H} = H \otimes I_d$ and H is the incidence matrix.

We now introduce a particularly important orthogonal projection matrix operator, which will be widely used in this paper. For any nonzero vector $x \in \mathbb{R}^d$ ($d \geq 2$), define the operator $P : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ as

$$P_x \triangleq I_d - \frac{x}{\|x\|} \frac{x^{\mathrm{T}}}{\|x\|}.$$

Note P_x is an orthogonal projection matrix which geometrically projects any vector onto the orthogonal compliment of x. It is easily verified that $P_x^{\mathrm{T}} = P_x$, $P_x^2 = P_x$, and P_x is positive semi-definite. Moreover, $\mathrm{Null}(P_x) = \mathrm{span}\{x\}$ and the eigenvalues of P_x are $\{0, 1, \ldots, 1\}$, where the zero eigenvalue is simple and the multiplicity of the eigenvalue 1 is $d - 1$.

In bearing rigidity theory, the relationship of two frameworks is evaluated by comparing the bearings of them. The bearings of two vectors are the same only if they are parallel to each other. As a result, the notion of parallel vectors is the core for the development of bearing rigidity theory. In our work, we use the projection matrix to characterize if two vectors in an arbitrary dimension are parallel to each other.

Corollary 1. *Two nonzero vectors $x, y \in \mathbb{R}^d$ ($d \geq 2$) are parallel if and only if $P_x y = 0$ (or equivalently $P_y x = 0$).*

Based on Corollary 1, we define the following notations of bearing rigidity analogously to the conventional distance rigidity theory.

Definition 1 (Bearing Equivalency)**.** *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are* bearing equivalent *if $P_{(p_i - p_j)}(p'_i - p'_j) = 0$ for all $(i,j) \in \mathcal{E}$.*

Definition 2 (Bearing Congruency)**.** *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are* bearing congruent *if $P_{(p_i - p_j)}(p'_i - p'_j) = 0$ for all $i, j \in \mathcal{V}$.*

Definition 3 (Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ is* bearing rigid *if there exists a constant $\epsilon > 0$ such that any framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$ and satisfies $\|p' - p\| < \epsilon$ is also bearing congruent to $\mathcal{G}(p)$.*

Definition 4 (Global Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ is* globally bearing rigid *if an arbitrary framework that is bearing equivalent to $\mathcal{G}(p)$ is also bearing congruent to $\mathcal{G}(p)$.*

We next define the bearing rigidity matrix and infinitesimal bearing rigidity. To do that, first define the *bearing function* $F_B : \mathbb{R}^{dn} \to \mathbb{R}^{dm}$ as

$$F_B(p) \triangleq \begin{bmatrix} \vdots \\ \frac{p_j - p_i}{\|p_j - p_i\|} \\ \vdots \end{bmatrix} \in \mathbb{R}^{dm}.$$

Each entry of $F_B(p)$ corresponds to the bearing of an edge in the framework. The *bearing rigidity matrix* is defined as the Jacobian of the bearing function,

$$R_B(p) \triangleq \frac{\partial F_B(p)}{\partial p} \in \mathbb{R}^{dm \times dn}. \tag{3}$$

Let $\mathrm{d}p$ be a variation of the configuration p. If $R_B(p)\mathrm{d}p = 0$, then $\mathrm{d}p$ is called an *infinitesimal bearing motion* of $\mathcal{G}(p)$. This is analogous to infinitesimal motions used in distance-based rigidity. Distance preserving motions of a framework include rigid-body translations and rotations, whereas bearing preserving motions of a framework include translations and scalings.

Definition 5 (Infinitesimal Bearing Rigidity)**.** *A framework is* infinitesimally bearing rigid *if the infinitesimal bearing motion only corresponds to translations and scalings of the entire framework.*

We next derive a useful matrix expression of $R_B(p)$.

Proposition 1. *The bearing rigidity matrix in* (3) *can be expressed as*

$$R_B(p) = \mathrm{diag}\left(\frac{P_{g_k}}{\|e_k\|}\right)\bar{H}. \tag{4}$$

Proof. Consider an oriented graph and express the bearings as $\{g_k\}_{k=1}^m$. Then, the bearing function can be written as $F_B(p) = [g_1^\mathrm{T}, \ldots, g_m^\mathrm{T}]^\mathrm{T}$. It follows from $g_k = e_k/\|e_k\|$ that

$$\frac{\partial g_k}{\partial e_k} = \frac{1}{\|e_k\|}\left(I_d - \frac{e_k}{\|e_k\|}\frac{e_k^\mathrm{T}}{\|e_k\|}\right) = \frac{1}{\|e_k\|}P_{g_k}.$$

As a result, $\partial F_B(p)/\partial e = \mathrm{diag}\left(P_{g_k}/\|e_k\|\right)$ and consequently

$$\frac{\partial F_B(p)}{\partial p} = \frac{\partial F_B(p)}{\partial e}\frac{\partial e}{\partial p} = \mathrm{diag}\left(\frac{P_{g_k}}{\|e_k\|}\right)\bar{H}.$$

□

The matrix expression (4) can be used to characterize the null space and the rank of the bearing rigidity matrix.

Proposition 2. *For any framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ ($n \geq 2$, $d \geq 2$), we have* $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R_B(p))$ *and consequently* $\mathrm{rank}(R_B(p)) \leq dn - d - 1$.

Proof. First, it is clear that $\mathrm{span}\{\mathbf{1} \otimes I_d\} \subseteq \mathrm{Null}(\bar{H}) \subseteq \mathrm{Null}(R_B(p))$. Second, since $P_{e_k}e_k = 0$, we have $R_B(p)p = \mathrm{diag}(P_{e_k}/\|e_k\|)\bar{H}p = \mathrm{diag}(P_{e_k}/\|e_k\|)e = 0$ and hence $p \subseteq \mathrm{Null}(R_B(p))$. The rank inequality $\mathrm{rank}(R_B(p)) \leq dn - d - 1$ follows immediately from $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R_B(p))$. □

For any undirected graph $\mathcal{G} = \{\mathcal{E}, \mathcal{V}\}$, denote $\mathcal{G}^\kappa$ as the complete graph over the same vertex set $\mathcal{V}$, and $R_B^\kappa(p)$ as the bearing rigidity matrix of the framework $G^\kappa(p)$. Then, the necessary and sufficient conditions for bearing equivalency, bearing congruency, and global bearing rigidity can be obtained as below.

Proposition 3. *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are bearing equivalent if and only if $R(p)p' = 0$, and bearing congruent if and only if $R^\kappa(p)p' = 0$.*

Proof. Since $R(p)p' = \operatorname{diag}(I_d/\|e_k\|)\operatorname{diag}(P_{g_k})\bar{H}p' = \operatorname{diag}(I_d/\|e_k\|)\operatorname{diag}(P_{g_k})e'$, we have

$$R(p)p' = 0 \Leftrightarrow P_{g_k}e'_k = 0,\ \forall k \in \{1, \ldots, m\}.$$

Therefore, by the definition of bearing equivalency, the two frameworks are bearing equivalent if and only if $R(p)p' = 0$. By the definition of bearing congruency, it can be analogously proved that two frameworks are bearing equivalent if and only if $R^\kappa(p)p' = 0$. $\square$

Theorem 1. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ ($n \geq 2, d \geq 2$) is globally bearing rigid if and only if $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$ or equivalently $\operatorname{rank}(R(p)) = \operatorname{rank}(R^\kappa(p))$.*

Proof. Necessity: Suppose framework $\mathcal{G}(p)$ is globally bearing rigid. We first prove $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. Consider an arbitrary $\mathrm{d}p$ satisfying $R(p)\mathrm{d}p = 0$. Since $R(p)p = 0$, we further have $R(p)(p + \mathrm{d}p) = 0$. As a result, framework $\mathcal{G}(p + \mathrm{d}p)$ is bearing equivalent to $\mathcal{G}(p)$ by Proposition 3. Then, it follows from the global bearing rigidity of $\mathcal{G}(p)$ that $\mathcal{G}(p + \mathrm{d}p)$ is also bearing congruent to $\mathcal{G}(p)$. By Proposition 3, we have $R^\kappa(p)(p + \mathrm{d}p) = 0$ and consequently $R^\kappa(p)\mathrm{d}p = 0$. Therefore, any $\mathrm{d}p \in \operatorname{Null}(R(p))$ is also in $\operatorname{Null}(R^\kappa(p))$ and thus $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. We second prove $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ holds for an arbitrary framework $\mathcal{G}(p)$ (even it is not bearing rigid). Consider an arbitrary $\mathrm{d}p$ satisfying $R^\kappa(p)\mathrm{d}p = 0$. As a result, $R^\kappa(p)(p + \mathrm{d}p) = 0$ and hence $\mathcal{G}(p + \mathrm{d}p)$ is bearing congruent to $\mathcal{G}(p)$ by Proposition 3. Since bearing congruency implies bearing equivalency, we know $R(p)(p + \mathrm{d}p) = 0$ and hence $R(p)\mathrm{d}p = 0$. Therefore, any $\mathrm{d}p \in \operatorname{Null}(R^\kappa(p))$ is also in $\operatorname{Null}(R(p))$ and thus $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$. In summary, $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$.

Sufficiency: Any framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$ satisfy $R(p)p' = 0$ by Proposition 3. Then, it follows from $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$ that $R^\kappa(p)p' = 0$, which means $\mathcal{G}(p')$ is also bearing congruent to $\mathcal{G}(p)$. As a result, $\mathcal{G}(p)$ is globally bearing rigid.

Because $R(p)$ and $R^\kappa(p)$ have the same column number, it follows immediately that $\operatorname{Null}(R^\kappa(p)) = \operatorname{Null}(R(p))$ if and only if $\operatorname{rank}(R^\kappa(p)) = \operatorname{rank}(R(p))$. $\square$

Theorem 2. *A framework is bearing rigid if and only if it is globally bearing rigid.*

Proof. It is obvious that global bearing rigidity implies bearing rigidity. We next prove the converse is also true. Suppose the framework $\mathcal{G}(p)$ is bearing rigid. By the definition of bearing rigidity and Proposition 3, any framework satisfying $R(p)p' = 0$ and $\|p' - p\| \leq \epsilon$ also satisfies $R^\kappa(p)p' = 0$. By denoting $\mathrm{d}p = p' - p$, we equivalently have

$$\forall \|\mathrm{d}p\| \leq \epsilon,\ R(p)(p + \mathrm{d}p) = 0 \Rightarrow R^\kappa(p)(p + \mathrm{d}p) = 0.$$

Then, it follows from $R(p)p = 0$ and $R^\kappa(p)p = 0$ that

$$\forall \|\mathrm{d}p\| \leq \epsilon,\ R(p)\mathrm{d}p = 0 \Rightarrow R^\kappa(p)\mathrm{d}p = 0,$$

which means $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. Since $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ for an arbitrary framework as shown in the proof of Theorem 1, we have $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$ and consequently $\mathcal{G}(p)$ is global bearing rigid by Theorem 1. $\square$

Figure 1: Collinear frameworks that are globally bearing rigid but *not* infinitesimally bearing rigid. Frameworks (a) and (b) are bearing equivalent and congruent and also globally bearing rigid. Observe, however, that the middle point can move along the line freely without changing any bearings, implying they are not infinitesimally bearing rigid.

By the definition, infinitesimal bearing rigidity implies bearing rigidity and thus global bearing rigidity by Theorem 2. But the converse is not true. Global bearing rigidity does not imply infinitesimal bearing rigidity in general. For example, the collinear frameworks shown in Fig. 1 are globally bearing rigid but not infinitesimally bearing rigid. From this example, we also know that global bearing rigidity does not imply a unique shape of the framework. In fact, as will be shown later, it is infinitesimal bearing rigidity that implies unique shapes.

We now give the necessary and sufficient condition for infinitesimal bearing rigidity.

Theorem 3. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ ($n \geq 2$, $d \geq 2$) is infinitesimally bearing rigid if and only if*

$$\mathrm{Null}(R_B(p)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p\} = \mathrm{span}\{\mathbf{1} \otimes I_d, p - \mathbf{1} \otimes \bar{p}\},$$

or equivalently $\mathrm{rank}(R_B(p)) = dn - d - 1$ *where* $\bar{p} = (\mathbf{1} \otimes I_d)^{\mathrm{T}} p/n$.

Proof. Proposition 2 shows $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R_B(p))$. Observe $\mathbf{1} \otimes I_d$ and p correspond to a rigid-body translation and a scaling of the framework, respectively. The stated condition directly follows from the definition of infinitesimal bearing rigidity. Note also that $\mathrm{span}\{\mathbf{1} \otimes I_d, p - \mathbf{1} \otimes \bar{p}\}$ is an orthogonal basis for $\mathrm{span}\{\mathbf{1} \otimes I_d, p\}$. □

In particular, a framework $\mathcal{G}(p)$ is infinitesimally bearing rigid in $\mathbb{R}^2$ if and only if $\mathrm{rank}(R_B(p)) = 2n - 3$, and in $\mathbb{R}^3$ if and only if $\mathrm{rank}(R_B(p)) = 3n - 4$. The next theorem shows that infinitesimal bearing rigidity can globally and uniquely determine the shape of a framework.

Theorem 4. *An infinitesimally bearing rigid framework can be globally and uniquely determined up to a translation and a scaling factor.*

Proof. Suppose $\mathcal{G}(p)$ is an infinitesimally bearing rigid framework in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Consider an arbitrary framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$. Our goal is to prove $\mathcal{G}(p')$ is different from $\mathcal{G}(p)$ only in a translation and a scaling factor.

Consider an oriented graph and denote the bearings of $\mathcal{G}(p)$ and $\mathcal{G}(p')$ as $\{g_k\}_{k=1}^m$ and $\{g'_k\}_{k=1}^m$, respectively. Then, it follows from the bearing equivalency that g_k is parallel to g'_k for all $k \in \{1, \ldots, m\}$. The configuration p' can always be decomposed as

$$p' = cp + \mathbf{1} \otimes \eta + q, \tag{5}$$

where $c \in \mathbb{R} \setminus \{0\}$ stands for a scaling factor, $\eta \in \mathbb{R}^d$ denotes a rigid-body translation of the framework, and $q \in \mathbb{R}^{dn}$, which satisfies $q \perp \mathrm{span}\{\mathbf{1} \otimes I_d, p\}$, represents a transformation other than translation and scaling. Note $\mathrm{Null}(R(p)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p\}$ due to the infinitesimal bearing rigidity of $\mathcal{G}(p)$. Then, multiplying $R(p)$ on both sides of (5) yields

$$R(p)p' = R(p)q, \tag{6}$$

Since $\mathcal{G}(p')$ is bearing equivalent to $\mathcal{G}(p)$, we have $R(p)p' = 0$ by Proposition 3. Therefore, (6) implies

$$R(p)q = 0.$$

Since $q \perp \text{span}\{\mathbf{1} \otimes I_d, p\} = \text{Null}(R(p))$, the above equation suggests $q = 0$. As a result, p' is different from p only in a scaling factor c and a rigid-body translation η. $\square$

III. Bearing-only Formation Control

In this section, we investigate bearing-only formation control of multi-agent systems in arbitrary dimensions in the presence of a global reference frame. It is assumed that each agent knows the orientation of a common (global) frame and the bearing measurements of their neighbors can be expressed in this frame. In practice, each agent may carry, for example, an inertial measurement unit (IMU) and a global positioning system (GPS) receiver to measure their three-dimensional orientations with respect to a global reference frame.

Consider n agents in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$) and assume there is a global inertial reference frame known to each agent. The vector quantities shown below are all expressed in this global frame. Denote $p_i \in \mathbb{R}^d$ as the position of agent $i \in \{1, \ldots, n\}$. The dynamics of agent i is

$$\dot{p}_i = v_i,$$

where $v_i \in \mathbb{R}^d$ is the velocity input to be designed. Denote $p = [p_1^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$ and $v = [v_1^{\mathrm{T}}, \ldots, v_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$. The underlying sensing graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is assumed to be undirected and fixed, and the formation is denoted by the framework $\mathcal{G}(p)$. The edge vector e_{ij} and the bearing g_{ij} are defined as in (1). Given an arbitrary orientation of the graph, we can express the edge and bearing vectors as $e = [e_1^{\mathrm{T}}, \ldots, e_m^{\mathrm{T}}]^{\mathrm{T}}$ and $g = [g_1^{\mathrm{T}}, \ldots, g_m^{\mathrm{T}}]^{\mathrm{T}}$ as defined in (2).

Since each agent knows the orientation of the global frame, the *bearing measurements* obtained by agent i are $\{g_{ij}\}_{j \in \mathcal{N}_i}$. The constant *bearing constraints* for the target formation are $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ with $g_{ij}^* = -g_{ji}^*$. Examples are given in Fig. 2 to illustrate bearing constraints.

Definition 6 (Feasible Bearing Constraints)**.** *The bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ are* feasible *if there exists a formation $\mathcal{G}(p)$ that satisfies $g_{ij} = g_{ij}^*$ for all $(i,j) \in \mathcal{E}$.*

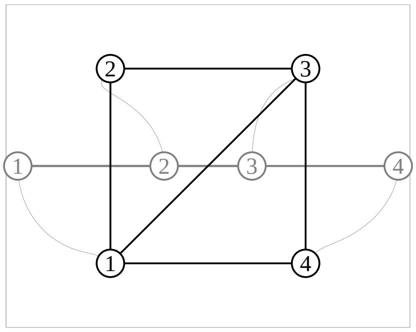
The bearing-only formation control problem to be solved in this section is stated as below.

Problem 1. *Given feasible constant bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ and an initial position $p(0)$, design v_i ($i \in \mathcal{V}$) based only on the bearing measurements $\{g_{ij}\}_{j\in\mathcal{N}_i}$ such that $g_{ij} \to g_{ij}^*$ as $t \to \infty$ for all $(i,j) \in \mathcal{E}$.*

A. A Bearing-Only Control Law

The proposed formation control law that relies only on bearing measurements is

$$v_i = -\sum_{j\in\mathcal{N}_i} P_{g_{ij}} g_{ij}^*, \quad \forall i \in \mathcal{V}. \tag{7}$$



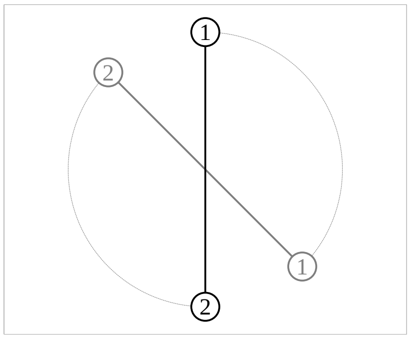


Figure 2: Initial formation: grey; target formation: black. (a) The bearing constraints satisfied by the target formation are $g_{12}^* = -g_{21}^* = [0,1]^{\mathrm{T}}$, $g_{23}^* = -g_{32}^* = [1,0]^{\mathrm{T}}$, $g_{34}^* = -g_{43}^* = [0,-1]^{\mathrm{T}}$, $g_{41}^* = -g_{14}^* = [-1,0]^{\mathrm{T}}$, and $g_{13}^* = -g_{31}^* = [\sqrt{2}/2, \sqrt{2}/2]^{\mathrm{T}}$. (b) The bearing constraints are $g_{12}^* = -g_{21}^* = [0,1]^{\mathrm{T}}$. As can be seen, the bearing error is reduced to zero while the inter-agent distance is unchanged.

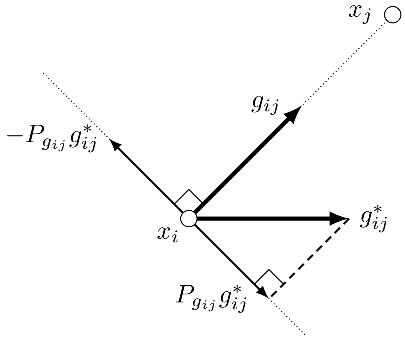


Figure 3: The geometric interpretation of control law (7). The control term $-P_{g_{ij}} g_{ij}^*$ is perpendicular to the bearing g_{ij}.

The control law has a clear geometric interpretation (see Fig. 3). Since the term $-P_{g_{ij}} g_{ij}^*$ is perpendicular to g_{ij} as $g_{ij}^{\mathrm{T}} P_{g_{ij}} g_{ij}^* = 0$, the control law attempts to reduce the bearing error of g_{ij} while preserving the distance between agents i and j. This geometric interpretation is also demonstrated by the example shown in Fig. 2(b).

The control law (7) can be equivalently expressed in a matrix-vector form. Since $g_{ij}^* = -g_{ji}^*$, the bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ can be reexpressed as $\{g_k^*\}_{k=1}^m$ by considering an oriented graph. Let $g^* = [(g_1^*)^{\mathrm{T}}, \ldots, (g_m^*)^{\mathrm{T}}]^{\mathrm{T}}$, then (7) can be written as

$$v = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \triangleq \mathcal{R}^{\mathrm{T}}(p) g^*. \tag{8}$$

It should be noted that the oriented graph is merely used to obtain the matrix expression while the underlying sensing graph of the formation is still the undirected graph $\mathcal{G}$. Moreover, it is worth mentioning that control law (8) is a modified gradient control law. If we consider the bearing error $\sum_{k=1}^m \|g_k - g_k^*\|^2$, a short calculation shows the corresponding gradient control law is $u = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}/\|e_k\|) g^*$, which is exactly $u = R_B^{\mathrm{T}}(p) g^*$, where $R_B(p)$ is the bearing rigidity matrix. However, the gradient control requires distance measurements $\|e_k\|$. By simply removing the distance term $\|e_k\|$, we can obtain the proposed control law (8).

We now examine certain properties of the control law. In particular, we show that both the centroid and scale of the formation are invariant quantities under the action of

(7). In this direction, define

$$\bar{p} \triangleq \frac{1}{n}\sum_{i=1}^{n} p_i,$$

to be the *centroid* of the formation, and

$$s \triangleq \sqrt{\frac{1}{n}\sum_{i=1}^{n} \|p_i - \bar{p}\|^2},$$

as the quadratic mean of the distances from the agents to the centroid. The quantity s can thus be interpreted as the *scale* of the formation. Since the inter-agent distances are not controlled, it is of great interest whether the scale of the formation will change under the proposed control law.

Proposition 4. *Under the control law* (8)*,*

$$\dot{p} \perp \operatorname{span}\{\mathbf{1} \otimes I_d, p\}.$$

Proof. The dynamics $\dot{p} = \mathcal{R}^{\mathrm{T}}(p)g^*$ implies $\dot{p} \in \mathrm{Range}(\mathcal{R}^{\mathrm{T}}(p))$. Since $\mathrm{Range}(\mathcal{R}^{\mathrm{T}}(p)) \perp \mathrm{Null}(\mathcal{R}(p))$, it follows that $\dot{p} \perp \mathrm{Null}(\mathcal{R}(p))$. Furthermore, $\mathrm{Null}(\mathcal{R}(p)) = \mathrm{Null}(R_B(p))$ and $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R_B(p))$ by Proposition 2 conclude the proof. □

Theorem 5. *The centroid $\bar{p}$ and the scale s of the formation are invariant under the control law* (8)*.*

Proof. Since $\bar{p} = (\mathbf{1}\otimes I_d)^{\mathrm{T}}p/n$, we have $\dot{\bar{p}} = (\mathbf{1}\otimes I_d)^{\mathrm{T}}\dot{p}/n$. It follows from $\dot{p} \perp \mathrm{Range}(\mathbf{1}\otimes I_d)$ as shown in Proposition 4 that $\dot{\bar{p}} \equiv 0$. Rewrite s as $s = \|p - \mathbf{1} \otimes \bar{p}\|/\sqrt{n}$. Then,

$$\dot{s} = \frac{1}{\sqrt{n}} \frac{(p - \mathbf{1} \otimes \bar{p})^{\mathrm{T}}}{\|p - \mathbf{1} \otimes \bar{p}\|}\dot{p}.$$

It follows from $\dot{p} \perp p$ and $\dot{p} \perp \mathbf{1} \otimes \bar{p}$ as shown in Proposition 4 that $\dot{s} \equiv 0$. □

The following results, which can be obtained from Theorem 5, characterize the behavior of the formation trajectories.

Corollary 2. *The formation trajectory under the control law* (8) *satisfies the following inequalities,*

(a) $s \leq \max_{i\in\mathcal{V}} \|p_i(t) - \bar{p}\| \leq s\sqrt{n-1}, \quad \forall t \geq 0.$

(b) $\|p_i(t) - p_j(t)\| \leq 2s\sqrt{n-1}, \quad \forall i, j \in \mathcal{V},\ \forall t \geq 0.$

Proof. First, we prove $\|p_i - \bar{p}\| \leq s\sqrt{n-1}$ for all $i \in \mathcal{V}$. On the one hand, invariance of the centroid implies that $\sum_{j\in\mathcal{V}}(p_j - \bar{p}) = (p_i - \bar{p}) + \sum_{j\in\mathcal{V}, j\neq i}(p_j - \bar{p}) = 0$, which implies

$$\|p_i - \bar{p}\|^2 \leq \left(\sum_{\substack{j\in\mathcal{V}\\ j\neq i}} \|p_j - \bar{p}\|\right)^2 \leq (n-1)\sum_{\substack{j\in\mathcal{V},\\ j\neq i}} \|p_j - \bar{p}\|^2. \tag{9}$$

On the other hand, scale invariance implies that $\|p_i - \bar{p}\|^2 + \sum_{j\in\mathcal{V}, j\neq i} \|p_j - \bar{p}\|^2 = ns^2$. Substituting this expression into (9) gives $\|p_i - \bar{p}\|^2 \leq (n-1)(ns^2 - \|p_i - \bar{p}\|^2)$, which indicates $\|p_i - \bar{p}\| \leq s\sqrt{n-1}$.

Second, we prove $s \leq \max_{i\in\mathcal{V}} \|p_i - \bar{p}\|$. Since $\max_{i\in\mathcal{V}} \|p_i - \bar{p}\|^2 \geq \|p_j - \bar{p}\|^2$, we have $n(\max_{i\in\mathcal{V}} \|p_i - \bar{p}\|^2) \geq \sum_{i=1}^{n} \|p_i - \bar{p}\|^2 = ns^2$, which implies $\max_{i\in\mathcal{V}} \|p_i - \bar{p}\| \geq s$.

Third, the inequality in (b) is obtained from $\|p_i(t) - p_j(t)\| = \|(p_i(t) - \bar{p}) - (p_j(t) - \bar{p})\| \leq \|p_i(t) - \bar{p}\| + \|p_j(t) - \bar{p}\| \leq 2s\sqrt{n-1}$. □

B. Formation Stability Analysis

In order to prove the formation stability, we adopt the following assumption.

Assumption 1. *A formation that satisfies the bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ is infinitesimally bearing rigid.*

Remark 1. *Assumption 1 means the bearing rigidity matrix $R_B(p) = \mathrm{diag}\left(I_d/\|e_k\|\right)\mathrm{diag}\left(g_k\right)\bar{H}$ and consequently $\mathcal{R}(p) = \mathrm{diag}\left(g_k\right)\bar{H}$ both have rank $dn - d - 1$.*

Definition 7 (Target Formation)**.** *Let $\mathcal{G}(p^*)$ be a* target formation *satisfying*

(a) Bearing: $(p_j^* - p_i^*)/\|p_j^* - p_i^*\| = g_{ij}^*$ for all $(i,j) \in \mathcal{E}$.

(b) Centroid: $\bar{p}^* = \bar{p}(0)$.

(c) Scale: $s^* = s(0)$.

Proposition 5. *The target formation $\mathcal{G}(p^*)$ in Definition 7 always exists and is unique under Assumption 1.*

Proof. The bearing constraints are feasible and the centroid and the scale of a formation can be changed continuously without affecting the bearings. Therefore, one can always find a formation that satisfies conditions (a), (b), and (c) in Definition 7, proving existence. Observe that $\mathcal{G}(p^*)$ is infinitesimally bearing rigid and hence it is uniquely determined up to a rigid-body translation and a scaling factor as shown in Theorem 4. The translation and the scale of $\mathcal{G}(p^*)$ are specified in conditions (b) and (c), showing the uniqueness of $\mathcal{G}(p^*)$. □

Let $\delta \triangleq p - p^*$. Denote $f_i(\delta) \triangleq -\sum_{j\in N_i} P_{g_{ij}} g_{ij}^*$ and $f(\delta) = [f_1^{\mathrm{T}}(\delta), \ldots, f_n^{\mathrm{T}}(\delta)]^{\mathrm{T}}$. The δ-dynamics can be expressed as

$$\dot{\delta} = f(\delta) = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^*. \tag{10}$$

As the bearing constraints are satisfied by $\mathcal{G}(p^*)$, showing that (8) solves Problem 1 is equivalent to showing the formation trajectory converges to $\mathcal{G}(p^*)$ (i.e., $\delta \to 0$ as $t \to \infty$). This idea was originally proposed in [24] to solve bearing-only formation control in two dimensions. A natural question that follows this idea is whether p^* can be calculated. In fact, it is easy to calculate p^* for simple formations with a small number of agents. For more complicated formations, the calculation of p^* may be nontrivial. But the calculation of p^* is not necessarily required to prove the convergence of the formation.

We next analyze the trajectories and equilibriums of the δ-dynamics (10). For the sake of simplicity, denote $r(t) \triangleq p(t) - (\mathbf{1} \otimes \bar{p})$ and $r^* \triangleq p^* - (\mathbf{1} \otimes \bar{p}^*)$. Due to the scale invariance, it can be verified that $\|r(t)\| \equiv \|r^*\| = \sqrt{n}s$ for all $t \geq 0$.

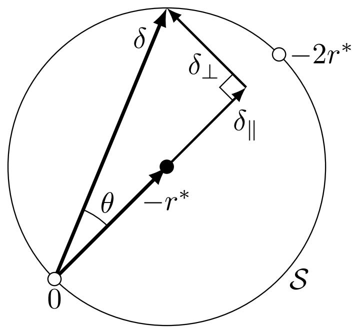


Figure 4: Geometric interpretation of δ, which satisfies $\|\delta + r^*\| = \|r^*\|$.

Proposition 6. *System* (10) *evolves on the surface of the sphere*

$$\mathcal{S} = \{\delta \in \mathbb{R}^{dn} : \|\delta + r^*\| = \|r^*\|\}.$$

Proof. Due to the centroid invariance, $\bar{p}(t) \equiv \bar{p}^*$. Then, it follows from $\delta(t) = p(t) - p^*$ that $\delta(t) + p^* - (\mathbf{1} \otimes \bar{p}^*) = p(t) - (\mathbf{1} \otimes \bar{p}) \Leftrightarrow \delta(t) + r^* = r(t)$. Due to the scale invariance, $\|r(t)\| \equiv \|r^*\| = \sqrt{n}s$. Then, $\|\delta(t) + r^*\| \equiv \|r^*\|$. $\square$

The state manifold $\mathcal{S}$ is illustrated by Fig. 4.

Lemma 1. *Given any two unit vectors $g_1, g_2 \in \mathbb{R}^d$, it always holds that $g_1^{\mathrm{T}} P_{g_2} g_1 = g_2^{\mathrm{T}} P_{g_1} g_2$.*

Proof. Since $g_1^{\mathrm{T}} g_1 = g_2^{\mathrm{T}} g_2 = 1$, we have $g_1^{\mathrm{T}} P_{g_2} g_1 = g_1^{\mathrm{T}}(I_d - g_2 g_2^{\mathrm{T}}) g_1 = g_1^{\mathrm{T}} g_1 - g_1^{\mathrm{T}} g_2 g_2^{\mathrm{T}} g_1 = g_2^{\mathrm{T}} g_2 - g_2^{\mathrm{T}} g_1 g_1^{\mathrm{T}} g_2 = g_2^{\mathrm{T}}(I_d - g_1 g_1^{\mathrm{T}}) g_2 = g_2^{\mathrm{T}} P_{g_1} g_2$. $\square$

Theorem 6. *Under Assumption 1, system* (10) *has two equilibrium points on $\mathcal{S}$,*

(a) $\delta = 0$, where $r = r^*$ and $g_{ij} = g_{ij}^*$, $\forall\,(i,j) \in \mathcal{E}$.

(b) $\delta = -2r^*$, where $r = -r^*$ and $g_{ij} = -g_{ij}^*$, $\forall\,(i,j) \in \mathcal{E}$.

Proof. Any equilibrium $\delta \in \mathcal{S}$ must satisfy $f(\delta) = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* = 0$, which implies

$$\begin{aligned} 0 &= (p^*)^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* = (e^*)^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \\ &= \sum_{k=1}^{m} (e_k^*)^{\mathrm{T}} P_{g_k} g_k^* = \sum_{k=1}^{m} \|e_k^*\| (g_k^*)^{\mathrm{T}} P_{g_k} g_k^*. \end{aligned}$$

Since $(g_k^*)^{\mathrm{T}} P_{g_k} g_k^* \geq 0$, the above equation implies $(g_k^*)^{\mathrm{T}} P_{g_k} g_k^* = 0$ for all k. As a result, by Lemma 1, we have $g_k^{\mathrm{T}} P_{g_k^*} g_k = 0$ and consequently $e_k^{\mathrm{T}} P_{g_k^*} e_k = 0$ for all k. Thus,

$$0 = e^{\mathrm{T}} \mathrm{diag}\left(P_{g_k^*}\right) e = p^{\mathrm{T}} \underbrace{\bar{H}^{\mathrm{T}} \mathrm{diag}\left(P_{g_k^*}\right)}_{\mathcal{R}^{\mathrm{T}}(p^*)} \underbrace{\mathrm{diag}\left(P_{g_k^*}\right) \bar{H}}_{\mathcal{R}(p^*)} p,$$

where the last equality uses the facts $P_{g_k^*} = P_{g_k^*}^2$ and $e = \bar{H} p$. The above equation indicates

$$\mathcal{R}(p^*) p = 0.$$

Observe $\mathcal{R}(p^*) = \mathrm{diag}(P_{g_k^*})\bar{H}$ has the same null space as the bearing rigidity matrix $R(p^*) = \mathrm{diag}(P_{g_k^*}/\|e_k^*\|)\bar{H}$. Since $\mathcal{G}(p^*)$ is infinitesimally bearing rigid by Assumption 1, it follows from Theorem 3 that $\mathrm{Null}(\mathcal{R}(p^*)) = \mathrm{span}\{\mathbf{1}\otimes I_d, p^* - \mathbf{1}\otimes\bar{p}^*\}$. Considering $\mathcal{R}(p^*)p = 0 \Leftrightarrow \mathcal{R}(p^*)(p-\mathbf{1}\otimes\bar{p}) = 0$, we have

$$p - \mathbf{1}\otimes\bar{p} \in \mathrm{span}\{\mathbf{1}\otimes I_d, p^* - \mathbf{1}\otimes\bar{p}^*\}.$$

Because $p - \mathbf{1}\otimes\bar{p} \perp \mathrm{Range}(\mathbf{1}\otimes I_d)$, we further know $p - \mathbf{1}\otimes\bar{p} \in \mathrm{span}\{p^* - \mathbf{1}\otimes\bar{p}^*\}$. Moreover, since $\|p - \mathbf{1}\otimes\bar{p}\| = \|p^* - \mathbf{1}\otimes\bar{p}^*\|$ due to the scale invariance, we have

$$p - \mathbf{1}\otimes\bar{p} = \pm(p^* - \mathbf{1}\otimes\bar{p}^*).$$

(i) In the case of $p - \mathbf{1}\otimes\bar{p} = p^* - \mathbf{1}\otimes\bar{p}^*$, we have $p = p^* \Leftrightarrow \delta = 0$ and consequently $g_{ij} = g_{ij}^*$ for all $(i,j)\in\mathcal{E}$. (ii) In the case of $p - \mathbf{1}\otimes\bar{p} = -(p^* - \mathbf{1}\otimes\bar{p}^*)$, we have $p = -p^* + 2(\mathbf{1}\otimes\bar{p}^*) \Leftrightarrow \delta = -2(p^* - \mathbf{1}\otimes\bar{p}^*)$, and consequently $g_{ij} = -g_{ij}^*$ for all $(i,j)\in\mathcal{E}$. □

Note the equilibrium $\delta = 0$ is desired, while the other one $\delta = -2r^*$ is undesired. At the undesired equilibrium, the formation $\mathcal{G}(p)$ is geometrically a point reflection of $\mathcal{G}(p^*)$ and bearing congruent to $\mathcal{G}(p^*)$.

Recall $\delta = p - p^*$. Choose the Lyapunov function as

$$V = \frac{1}{2}\|\delta\|^2$$

Based on this Lyapunov function, we next prove the almost global exponential stability of the desired equilibrium $\delta = 0$.

Theorem 7 (Almost Global Exponential Stability)**.** *Under Assumption 1, the system trajectory $\delta(t)$ of* (10) *exponentially converges to $\delta = 0$ from any $\delta(0)\in\mathcal{S}$ except $\delta(0) = -2r^*$.*

Remark 2. *In terms of bearings, Theorem 7 indicates that $g_{ij}(t)$ converges to g_{ij}^* for all $(i,j)\in\mathcal{E}$ from any initial conditions except $g_{ij}(0) = -g_{ij}^*, \forall(i,j)\in\mathcal{E}$.*

Proof. The derivative of V is $\dot{V} = \delta^{\mathrm{T}}\dot{\delta} = (p-p^*)^{\mathrm{T}}\dot{p} = -(p^*)^{\mathrm{T}}\dot{p}$. Substituting control law (8) into $\dot{V}$ yields

$$\begin{aligned}\dot{V} &= -(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^* = -(e^*)^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^* \\ &= -\sum_{k=1}^{m}(e_k^*)^{\mathrm{T}}P_{g_k}g_k^* = -\sum_{k=1}^{m}\|e_k^*\|(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* \le 0. \end{aligned} \tag{11}$$

Since $\dot{V}\le 0$, we have $\|\delta(t)\|\le\|\delta(0)\|$ for all $t\ge 0$. Furthermore, it follows from Lemma 1 that

$$(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* = g_k^{\mathrm{T}}P_{g_k^*}g_k,$$

substituting which into (11) gives

$$\begin{aligned}\dot{V} &= -\sum_{k=1}^{m}\|e_k^*\|g_k^{\mathrm{T}}P_{g_k^*}g_k = -\sum_{k=1}^{m}\frac{\|e_k^*\|}{\|e_k\|^2}e_k^{\mathrm{T}}P_{g_k^*}e_k \\ &\le -\underbrace{\frac{\min_{k=1,\dots,m}\|e_k^*\|}{4(n-1)s^2}}_{\alpha}\sum_{k=1}^{m}e_k^{\mathrm{T}}P_{g_k^*}e_k, \end{aligned} \tag{12}$$

where the inequality uses the fact $\|e_k\| \leq 2\sqrt{n-1}s$ by Corollary 2(b). Inequality (12) can be further written as

$$
\begin{aligned}
\dot{V} &\leq -\alpha e^{\mathrm{T}} \mathrm{diag}(P_{g_k^*}) e = -\alpha p^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k^*}) \bar{H} p \\
&= -\alpha \delta^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k^*}) \bar{H} \delta \quad \big(\text{Due to } \mathrm{diag}(P_{g_k^*}) \bar{H} p^* = 0\big) \\
&= -\alpha \delta^{\mathrm{T}} \underbrace{\bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k^*})}_{\mathcal{R}^{\mathrm{T}}(p^*)} \underbrace{\mathrm{diag}(P_{g_k^*}) \bar{H}}_{\mathcal{R}(p^*)} \delta.
\end{aligned} \tag{13}
$$

Observe $\mathcal{R}(p^*)$ has the same rank and null space as the bearing rigidity matrix $R_B(p^*)$. Under the assumption of infinitesimal bearing rigidity, it follows from Theorem 3 that $\mathrm{Null}(\mathcal{R}(p^*)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p^*\}$ and $\mathrm{rank}(\mathcal{R}(p^*)) = dn - d - 1$. As a result, the smallest $d+1$ eigenvalues of $\mathcal{R}^{\mathrm{T}}(p^*)\mathcal{R}(p^*)$ are zero. Let the minimum positive eigenvalue of $\mathcal{R}^{\mathrm{T}}(p^*)\mathcal{R}(p^*)$ be λ_{d+2}. Decompose δ to $\delta = \delta_\perp + \delta_\parallel$, where $\delta_\perp \perp \mathrm{Null}(\mathcal{R}(p^*))$ and $\delta_\parallel \in \mathrm{Null}(\mathcal{R}(p^*))$. Then (13) becomes

$$
\dot{V} \leq -\alpha \lambda_{d+2} \|\delta_\perp\|^2. \tag{14}
$$

Note $\delta_\parallel$ is the orthogonal projection of δ on $\mathrm{Null}(\mathcal{R}(p^*)) = \mathrm{span}\{\mathbf{1} \otimes I_d, r^*\}$. Because $\delta \perp \mathrm{span}\{\mathbf{1} \otimes I_d\}$, we know $\delta_\parallel$ actually is the orthogonal projection of δ on r^* (see Fig. 4). Let θ be the angle between δ and $-r^*$. Thus, $\|\delta_\perp\| = \|\delta\| \sin\theta$, and (14) becomes

$$
\dot{V} \leq -\alpha \lambda_{d+2} \sin^2\theta \|\delta\|^2. \tag{15}
$$

It can be seen from Fig. 4 that $\theta \in [0, \pi/2)$. Let θ_0 be the value of θ at time $t = 0$. Since $\|\delta(t)\| \leq \|\delta(0)\|$ for all t, it is clear from Fig. 4 that $\theta(t) \geq \theta_0$. Then, (15) becomes

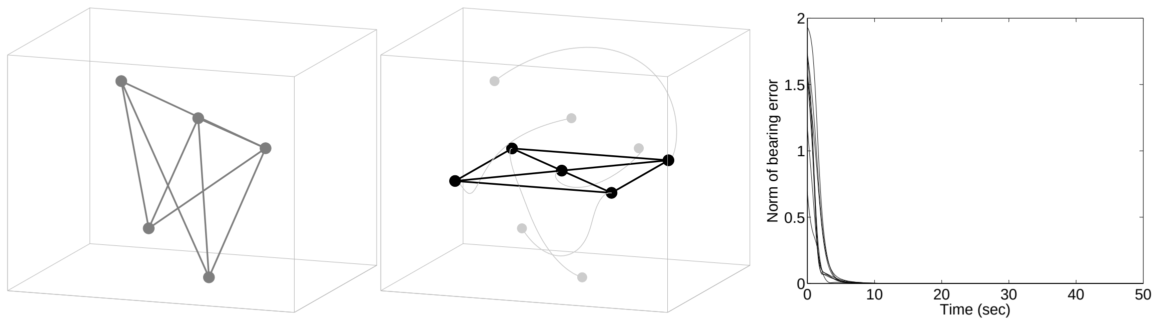
$$
\dot{V} \leq -\underbrace{2\alpha \lambda_{d+2} \sin^2\theta_0}_{K} V.
$$

(i) If $\theta_0 > 0$, then $K > 0$. As a result, the error $\|\delta(t)\|$ decreases to zero exponentially fast. (ii) If $\theta_0 = 0$, it can be seen from Fig. 4 that $\delta(0) = -2r^*$ which is the undesired equilibrium. In summary, the system trajectory $\delta(t)$ converges to $\delta = 0$ exponentially fast from any initial points except $\delta = -2r^*$. □

We would like to mention that the eigenvalue λ_{d+2} of $\mathcal{R}^{\mathrm{T}}(p^*)\mathcal{R}(p^*)$ affect the convergence rate of the system. Motivated by the distance rigidity maintenance control [31], we call λ_{d+2} as the *bearing rigidity eigenvalue*. It is obvious that $\lambda_{d+2} > 0$ if and only if $\mathcal{G}(p^*)$ is infinitesimally bearing rigid. As a result, λ_{d+2} can be viewed as a measure of the "degree of infinitesimal bearing rigidity" of a framework.

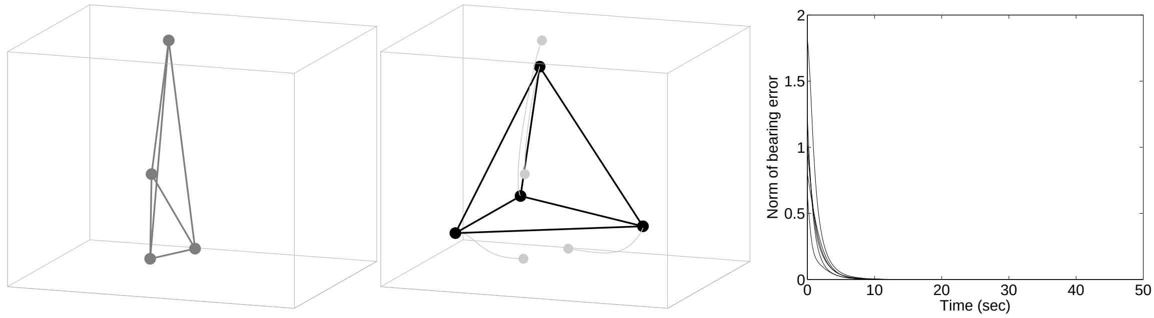
IV. Simulation

We have already presented two simulation examples previously in Fig. 2. As shown in Fig. 2(a), the collinear initial formation is not a problem for bearing-only formation control though they may cause troubles in distance formation control. More simulation examples are shown in Figs. 5, 6, and 7. The initial formations in these examples are generated randomly. Figure 5 is motivated by the spacecraft interferometry problem. In [6] it is shown that the so-called *X-array* aperture configuration provides the best performance. Using the proposed bearing-only formation control, a random non-coplanar configuration is able to converge to the desired X-array formation. In [4], tetrahedral formation shapes are considered motivated by ongoing formation spacecraft missions. As shown in Figs. 5 and 6, these formations can be achieved efficiently under the action of the proposed control law. Finally, Figure 7 shows the proposed control for a large formation consisting of 27 agents with the target formation a grid in three-dimensional space. In all examples the bearing error is also plotted.



(a) Initial formation (b) Trajectory (c) Norm of bearing error

Figure 5: A three-dimensional coplanar formation with $n = 5$, $m = 8$, and $\text{rank}(R_B) = 11 = 3n - 4$.

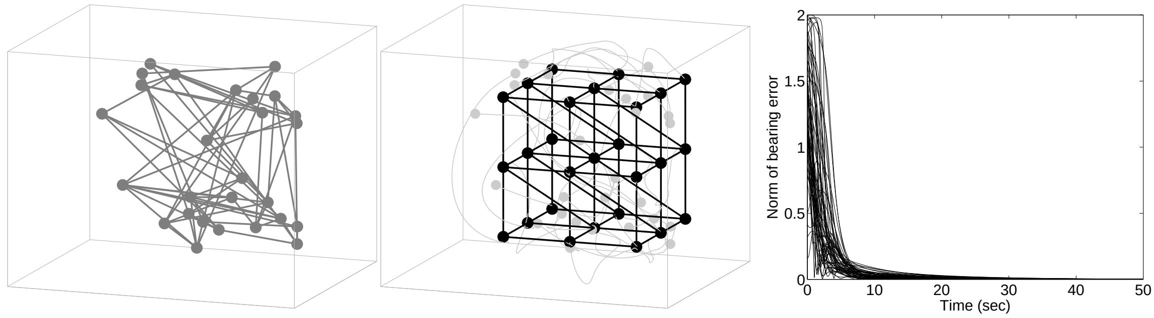


(a) Initial formation (b) Trajectory (c) Norm of bearing error

Figure 6: A three-dimensional tetrahedral formation with $n = 4$, $m = 6$, and $\text{rank}(R_B) = 8 = 3n - 4$.

V. Conclusions

In this paper, we first proposed a bearing rigidity theory that is applicable to arbitrary dimensional spaces. We then proposed a bearing-only formation control law and proved the almost global and exponential stability for infinitesimal bearing rigid formations. In recent years, distance-based control is the most widely adopted approach to formation shape control problems. The bearing rigidity and bearing-based formation control proposed in this paper provide an attractive alternative to the distance-based approaches. For example, distance-based formation control laws usually can only ensure local stability, while the proposed bearing-based control law ensures almost global stability. Moreover, distance-based formation control usually relies on the assumption on minimal infinitesimal distance rigidity. But the infinitesimal distance rigidity does not uniquely determine the shape of a framework. As a result, the formation may converge to an undesired shape given certain initial conditions. In addition, the minimal rigidity assumption also con-



(a) Initial formation (b) Trajectory (c) Norm of bearing error

Figure 7: A three-dimensional grid formation with $n = 27$, $m = 62$, and $\text{rank}(R_B) = 77 = 3n - 4$.

strains the number of the edges in the formation and consequently restricts its application in practice. As a comparison, the bearing-based formation control relies on infinitesimal bearing rigidity, which can uniquely determine the shape of a framework and does not constrain the edge number.

Acknowledgements

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