Structural Stability of Linear Time-Invariant Systems

Graph Theory in Systems and Controls: part 2

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Which structured LTI systems can sustain stable dynamics?

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{32} & 0 \\ 0 & a_{42}0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} u$$

- Does there exist values of the a_{ij}'s that yield asymptotically stable dynamics? If so, we call the system structurally stable.
- Does there exist values of the a_{ij}'s and b_i's that yield controllable dynamics? If so, we call the system structurally controllable.
- Recall: Linear time-invariant dynamics is asymptotically stable iff the eigevalues of the system matrix have strictly negative real parts.
- Graph theory is the natural framework to study structural stability.

Reformulating the structural stability problem

$$A = \begin{bmatrix} 0 & * & * & 0 & * \\ * & * & 0 & * & * \\ 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

$*$ entries are arbitrary real

0 entries are fixed to zero

Definition (Zero-pattern (ZP))

Set E_{ij} to be the $n \times n$ matrix with all entries 0 except for the ijth one, which is 1. We call a *zero pattern* a vector space $\mathcal{Z}$ of matrices

$$A = \sum_{(i,j)\in\mathcal{N}} a_{ij}E_{ij}.$$

- Does the ZP contain stable (Hurwitz) matrices?
- We call a ZP that contains Hurwitz matrices stable

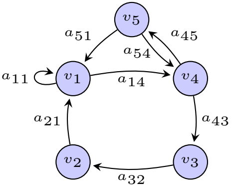
Hurwitz Digraphs and Zero-Patterns

- Think of a ZP as an adjacency matrix with

$$0 \longrightarrow 0$$
$$* \longrightarrow 1$$

- There is a bijection between zero patterns $\mathcal{Z}$ and digraphs $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \mathcal{N}$.

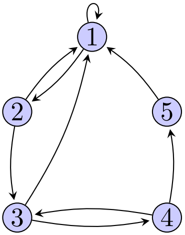
$$\begin{bmatrix} * & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

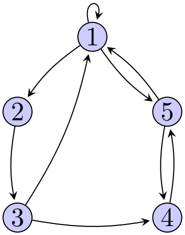


- We call a graph Hurwitz or stable if the corresponding ZP is stable.

How to determine if a graph is Hurwitz? How to create Hurwitz graphs?

Which graph is stable?





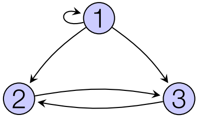
$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

Which graph is stable?

Key idea: need enough mixing of information

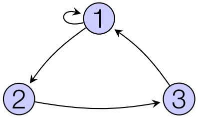
Lemma

A digraph G is stable only if every strongly connected component has a node with a self-loop



Not stable: the strongly connected component $\{2, 3\}$ has no nodes with a self-loop.

This is not the end of the story...

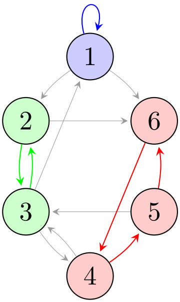


The graph is strongly connected and has a self-loop, yet not stable.

$\longrightarrow$ need to find the graphical structure that enables stability

k-decompositions

- k-cycle in G: a sequence of k distinct nodes connected by edges.
- Two cycles are disjoint if they have no nodes in common.
- k-decomposition in G: union of *disjoint cycles* covering k nodes.
  A k-decomposition is given by cycles $S_1, \ldots, S_l$ if the S_i are disjoint and $|S_1| + \cdots + |S_l| = k$.
- Hamiltonian cycle (resp. decomposition): n-cycle (resp. decomposition).

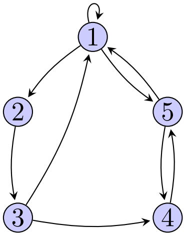


1-cycle = (1)
2-cycle: (23)
3-cycle: (456)
3-decomp.: $(1)(23)$ or (456)
4-decomp.: $(1)(456)$
5-decomp.: $(23)(456)$

A necessary condition for stability

Theorem[1]

A digraph G is stable only if it contains a k-decomposition for each $k = 1, 2, \ldots, n$



$$\begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

1-decomp.: (1), 2-decomp.: (15), 3-decomp.:$(1)(45)$ but no 4-decomp. $\longrightarrow$ not stable.

[1] B. “Sparse Stable Systems”, Systems and Control Letters, 2013

A necessary condition for stability: sketch of proof

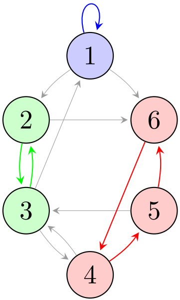
- S_k: symmetric group on k characters.
- For $\sigma \in S_k$, let $\sigma(i)$ be the position of the ith in the permutation.

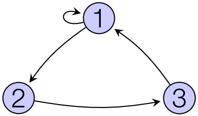
  e.g. $\sigma : \{1,2,3,4\} \to \{2,1,4,3\}$ then $\sigma(1) = 2$ and $\sigma(3) = 4$.
- It is known that A is Hurwitz only if all coefficients of its characteristic polynomial are non-zero.
- Characteristic polynomial of A is given by

$$\det(I\lambda - A) = \sum_{k=0}^{n-1} (-1)^k \lambda^k \sum_{\sigma \in S_{n-k}} (-1)^\sigma \prod_{i=1}^{n-k} a_{i,\sigma(i)}$$

A necessary condition for stability: sketch of proof (II)

- Each term $\prod_{i=1}^{k} a_{i,\sigma(i)}$ corresponds to a k-decomposition.
- Said otherwise: each permutation in S_k corresponds to a k-decomposition: e.g. permutation in S_3 that sends $\{4,5,6\}$ to $\{5,6,4\}$ is depicted in red. permutation in S_3 that sends $\{1,2,3\}$ to $\{1,3,2\}$ is depicted in blue+green.
- Conclusion: no k-decompositions $\Longrightarrow$ degree $n-k$ term in characteristic polynomial of any matrix in $\mathcal{Z}$ is zero $\Longrightarrow$ graph and ZP are not stable





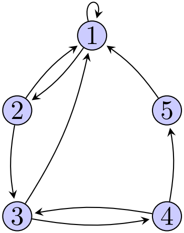
$$p(s) = s^3 - a_{11}s^2 + 0s - a_{12}a_{23}a_{31}.$$

A sufficient condition for stability

Theorem[2]

A digraph G is stable if it contains a sequence of *nested* k-decomposition for each $k = 1, 2, \dots, n$.

We say that a k-decomposition K_1 is nested in K_2 if the node set of K_1 is included in the one of K_2



$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & 0 & 0 & 0 & 0 \end{bmatrix}$$

1-decomp.: (1), 2-decomp.: (12), 3-decomp.:(123), 4-decomp.:$(12(34)$, 5-decomp.:(12345).

[2] B. “Sparse Stable Systems”, Systems and Control Letters, 2013

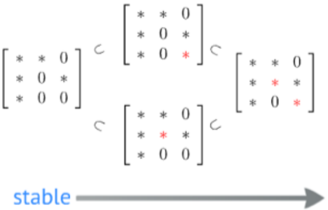
Are the necessary and sufficient conditions close?

- There are many graphs that are stable, but do not pass the sufficient condition.
- From our simulations, we observe that the necessary condition is close to being sufficient: the number of graphs that pass the necessary condition and are *not* stable is relatively small.
- Stability is not generic. The proportion of stable matrices in a ZP can be *very small*.
- Hence simulations studies are “hard”: one needs to sample many matrices in a SMS to conclude non-stability. Very unlike structural controllability: almost all systems in a zero-pattern are controllable. Sample one system: with probability one, it is controllable if the zero pattern is.

Minimal stable graphs and notions of robustness

Observation: adding an edge to a stable graph yields another stable graph.

We say that graph stability is monotone with respect to edge addition.

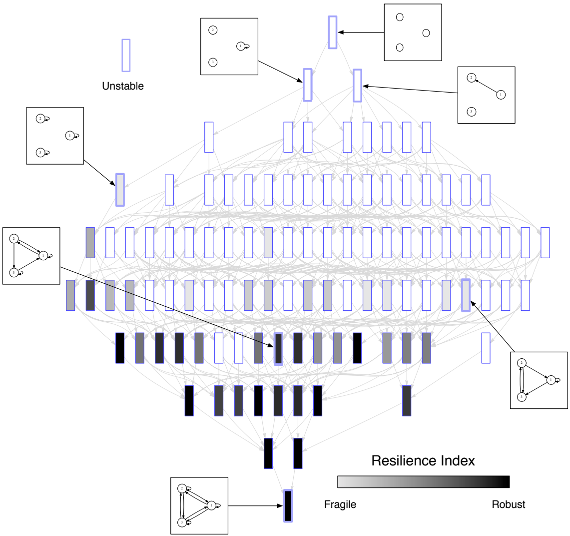


This simple observation yields two interesting definitions:

- Minimal stable graphs: stable graphs for which removing *any* edge yields an *unstable* graph.
  All stable graphs are "descendants" of minimal stable graphs. We can think of them as "prime" graphs.

- Robustly stable graphs: stable graphs for which removing *any* edge yields a *stable* graph.

The Tree of Three-Graphs



Box $\to$ graph on three nodes

Same # edges $\to$ same row

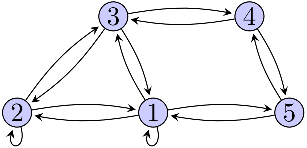
Edge between box denotes inclusion

Shade: $\dfrac{\text{\# stable ancestors}}{\text{\# ancestors}}$

Minimal stable: lightest shade. There are 7.

Reciprocal or Symmetric graphs

- It is often the case that information exchange is *bilateral*: $i \leftrightarrow j$.
- We call a graph reciprocal or symmetric of to every edge $(i, j) \in E$ there is an edge $(j, i) \in E$.
- The corresponding ZP is symmetric:



$$A = \begin{bmatrix} * & * & * & 0 & * \\ * & * & * & 0 & 0 \\ * & * & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

- Two cases: either the matrices in the ZP are symmetric (*strongly symmetric ZP*) or not necessarily symmetric (*weakly symmetric ZP*).

Stability of Symmetric Graphs

Definition[3]

A ZP is *(weakly symmetric* if to a free variable in position ij corresponds a free variable in position ji. A ZP is *strongly symmetric* if it only contains symmetric matrices.

Theorem[3]

A strongly symmetric ZP is stable *if and only if* all its diagonal elements are free.

Theorem[3]

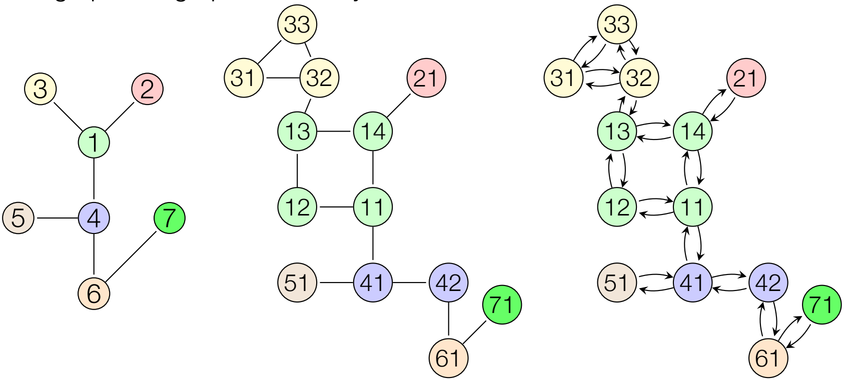
A weakly symmetric ZP is stable *if and only if* its graph is so that

1. Every node is strongly connected to a self-loop
2. The graph contains a Hamiltonian decomposition.

[3] A. Kirkoryan and B. "Symmetric Sparse Systems", CDC 2014.

Key notion: fat trees

The proof of the last theorem is graphical in nature. We sketch is here.
A tree graph is a graph without cycles.



- Tree graph → Nodes can be cycles → Edges are symmetric → fat tree

Stability of symmetric graphs

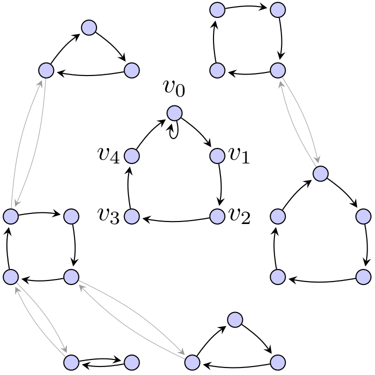
- Proof idea: Given a symmetric graph G, show that if
  1. Every node in G is connected to a self-loop
  2. G contains a Hamiltonian decomposition

  $\rightarrow$ then there exists a sequence of *nested* k-decompositions, $k = 1, \ldots, n$.

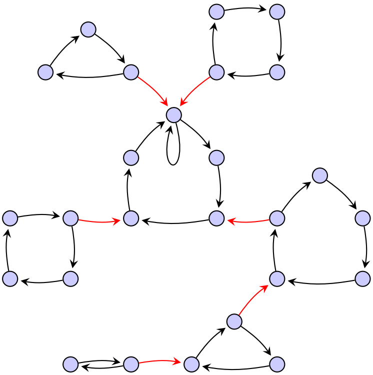
- The conclusion above says that we satisfy the sufficient condition presented earlier.

- Proof technique: find a fat tree in G. Fat trees provide a natural ordering of nodes. Use the ordering to exhibit nested k-decompositions:
  We label (order) the nodes so that $\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, \ldots, n\}$ all have k-decompositions. By construction, they are nested.

Stability of symmetric graphs (II)

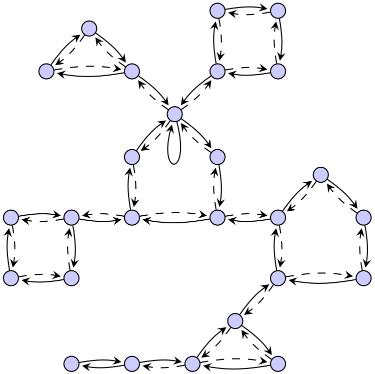


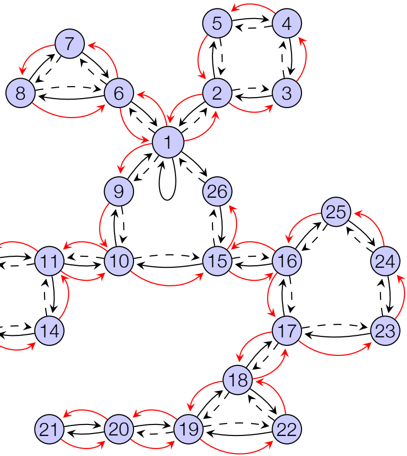
Draw the cycles of a Hamiltonian decomposition of G. This is a subgraph of G.



Connect every cycle to the cycle with the self-loop. We can do so by assumption 1.

Stability of symmetric graphs (III)



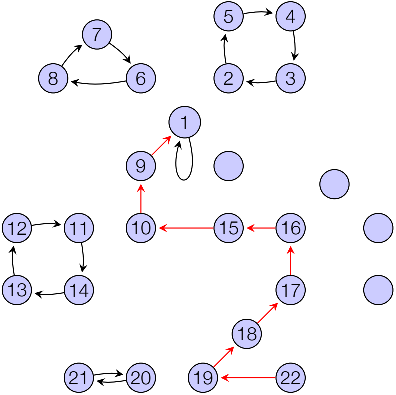


Add reciprocal edges. The resulting graph is a planar subgraph of G by construction.

Ordering: Set v_0 at 1. Order nodes counter-clockwise. Skip already numbered nodes. By construction, no node lies inside $\rightarrow$ complete ordering. Call this graph P.

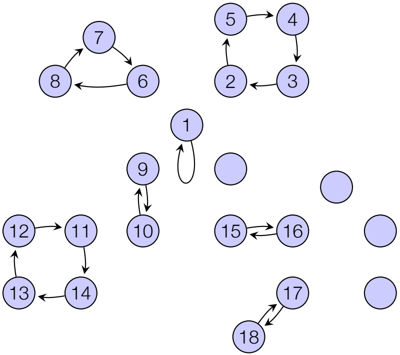
Stability of symmetric graphs: (IV)

The last graph shown is a subgraph of G. We show that is satisfies the hypothesis of Theorem 2.



- There is a unique path from any node k to 1 using the plain edges of P only.
- Key observation: by construction, the subgraph induced by the node set $\{1, 2, \ldots, k\}$ is the union of the path joining 1 to k and l-cycles.

Stability of symmetric graphs: (V)



A $n = 22$-decomposition

- The subgraph induced by nodes $\{1, \ldots, k\}$ admits a Hamiltonian decomposition, which is thus a k-decomposition of G.
- Depending on whether the path joining 1 to k has an even or odd number of nodes, the decomposition is in 2-cycles (even) or self=loop+2 cycles (odd).
- Repeating the procedure for each node $k = 1, \ldots, n$, we obtain nested k-decompositions.

Structural Stability of Random Graphs

- Random graph theory provides a different lens to look at what may otherwise be hard problems.
- We look for conditions under which a sample graph form a given distribution is structurally stable with overwhelming probability.
- The results are asymptotic in the number of nodes.
- Allows us to overlook finer structural details and obtain answers when the graph is very large.
- Recall: Bernoulli distribution with parameter p: $P(\omega = 1) = p$ $P(\omega = 0) = 1 - p$, $\omega \in \Omega = \{0, 1\}$.

Random graphs models

- We look at two random graph models for symmetric ZP
- Model 1: variable number of edges $\mathcal{G}^n_{p,q}$
  1. Graph on n nodes
  2. Existence of an edge between nodes i and j, $i \neq j$ are independent Bernoulli random variables with parameter p
  3. Existence of a self-loop are independent Bernoulli r.v. with parameter q.
- Model 2: fixed number of edges $\mathcal{F}^n_{M,K}$
  1. Graph on n nodes
  2. Exactly M edges (i, j), chosen uniformly at random amongst all possible edges (i, j), $i \neq j$.
  3. Exactly K self-loops chosen uniformly at random.

Definition

We say that **almost every** random graph G^n has a property X, if $\mathbb{P}(G^n \text{ has } X) \to 1$ as $n \to \infty$

Problem Statement

- We consider probabilities that depend on n. We need $p(n), q(n) \to 0$ $n \to \infty$, otherwise random graphs a very dense.

Problem

For what magnitudes of $p = p(n)$ and $q = q(n)$, is almost every random graph $\mathcal{G}^n_{p,q}$) stable? For what magnitudes of $M = M(n)$ and $K = K(n)$, is almost every random graph $\mathcal{F}^n_{M,K}$) stable?

- Define ω_1, ω_2, such that:

$$p = p(n) = \frac{\ln(n) + \omega_1}{n}, \quad q = q(n) = \frac{\omega_2}{n}.$$

This particular form for $p(n)$, $q(n)$ makes statements easier.

Results for Model 1

Theorem[4]

Assume that $q(n) < 1 - \varepsilon$ for some $\varepsilon > 0$

1. Almost every graph in $\mathcal{G}^n_{0,q}$ contains a self-loop *if and only if* $\omega_2 \to \infty$.
2. Almost every graph in $\mathcal{G}^n_{p,0}$ contains a Hamiltonian decomposition *if and only if* $\omega_1 \to \infty$.
3. $$\mathbb{P}(\mathcal{G}^n_{p,q} \text{ is stable}) \to 1 \quad \Longleftrightarrow \quad \omega_1, \omega_2 \to \infty.$$

$$p = p(n) = \frac{\ln(n) + \omega_1}{n}, \quad q = q(n) = \frac{\omega_2}{n}.$$

p is probability of an edge, q is probability of a self loop

[4] B., A. Kirkoryan, preprint; A. Kirkoryan PhD thesis

Results for Model 2

Define ω_1, ω_2 such that:

$$M = M(n) = \frac{n(\ln(n) + \omega_1)}{2}, \quad K = K(n) = \omega_2.$$

Theorem[5]

Assume that $M < \frac{n^2(1-\varepsilon)}{2}$ for some $\varepsilon > 0$, then

$$\mathbb{P}(\mathcal{G}^n_{M,K} \text{ is stable}) \to 1 \quad \Longleftrightarrow \quad \omega_1 \to \infty, \;\; \omega_2 \geq 1.$$

[5] B., A. Kirkoryan, preprint; A. Kirkoryan PhD thesis

Thank you for your attention!