Graph Theory in Systems and Control

Graphs: Unexplored Opportunities

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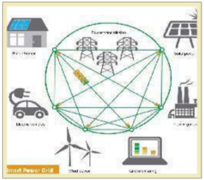
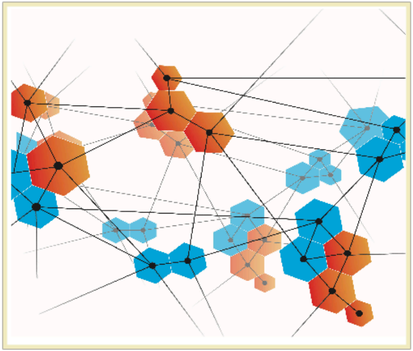
Networked Dynamic Systems



NETWORKS OF DYNAMICAL SYSTEMS ARE ONE OF THE ENABLING TECHNOLOGIES OF THE FUTURE



Graphs at CDC

Why do we need this tutorial?

Network analysis and control
MoA03.5, MoA05.6, MoA07.2, MoA09.6, MoA12.2, MoA12.6, MoB03.3, MoB05.3, MoB10.6, MoB12.1, MoC03.3, MoC03.4, MoC06.1, MoC06.2, MoC13.6, MoC14.3, TuA03.6, TuA04.3, TuB09.1, TuB09.2, TuB12.1, TuB12.2, TuB12.3, TuB12.4, TuB12.5, TuB12.6, TuB16.3, TuB16.4, TuC04.3, TuC04.6, TuC05.1, TuC09.1, TuC09.3, TuC10.5, TuC12.1, TuC12.2, TuC12.3, TuC12.4, TuC12.5, TuC12.6, TuC18.5, TuC18.6, WeA09.2, WeA09.3, WeA09.4, WeA09.5, WeA09.6, WeA12.1, WeA12.2, WeA12.3, WeA12.4, WeA12.5, WeA12.6, WeB03.6, WeB05.1, WeB05.4, WeB12.5, WeB12.6, WeB13.1, WeB13.2, WeB13.3, WeB13.4, WeB14.1, WeB14.6, WeB18.6, WeC01.6, WeC05.1, WeC05.5, WeC12.1, WeC12.3

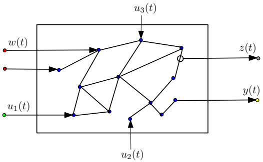
Networked control systems
MoA01.3, MoA03.4, MoA03.5, MoA04.2, MoA04.3, MoA04.4, MoA04.5, MoA04.6, MoA05.1, MoA05.3, MoA10.5, MoA12.1, MoA12.2, MoA12.3, MoA12.4, MoA12.5, MoA12.6, MoB03.2, MoB04.3, MoB04.4, MoB04.6, MoB09.4, MoB12.1, MoB12.2, MoB12.3, MoB12.4, MoB12.5, MoB12.6, MoB14.4, MoC03.4, MoC04.2, MoC04.3, MoC04.5, MoC09.4, MoC10.5, MoC12.1, MoC12.2, MoC12.3, MoC12.4, MoC12.5, MoC12.6, MoC13.1, MoC13.2, MoC13.4, MoC13.5, MoC18.4, MoC19.4, TuA01.6, TuA05.2, TuA10.1, TuA12.1, TuA12.2, TuA12.3, TuA12.4, TuA12.5, TuA12.6, TuA15.5, TuA21.3, TuB02.4, TuB04.4, TuB07.2, TuB12.4, TuB12.5, TuB14.3, TuB14.4, TuB19.3, TuC02.4, TuC03.2, TuC05.3, TuC05.5, TuC09.1, TuC11.6, TuC13.1, TuC16.4, WeA03.1, WeA03.4, WeA03.5, WeA05.6, WeA09.3, WeA09.5, WeA12.4, WeA12.5, WeA12.6, WeA14.4, WeA16.1, WeB01.3, WeB04.3, WeB04.4, WeB04.5, WeB08.1, WeB09.5, WeB10.3, WeB10.4, WeB12.3, WeB12.4, WeB13.1, WeB18.1, WeB19.3, WeB19.4, WeC07.2, WeC15.6, WeC17.3, WeC20.4, WeC21.1, WeC21.4

Control system architecture
MoA17.2, MoC07.6, TuA04.5, TuB06.3, TuB12.6, WeA06.2, WeB14.3, WeC05.4
See also Large-scale Systems

Cooperative control
MoA03.3, MoA03.4, MoA03.6, MoA11.6, MoA14.1, MoA14.2, MoA14.3, MoA14.4, MoA14.5, MoB03.1, MoB03.4, MoB05.5, MoB12.6, MoB14.1, MoB14.2, MoB14.3, MoB14.4, MoB14.5, MoB16.5, MoB17.2, MoB17.4, MoC03.6, MoC12.2, MoC14.1, MoC14.2, MoC14.3, MoC14.4, MoC14.5, MoC14.6, MoC17.2, MoSP1.1, TuA03.2, TuA03.3, TuA05.2, TuA09.6, TuA10.6, TuA11.1, TuA12.1, TuA14.4, TuA16.5, TuB04.1, TuB14.5, TuB17.6, TuC05.1, TuC09.2, TuC11.5, TuC11.6, TuC14.2, TuC14.3, WeA03.5, WeA05.2, WeA05.4, WeA05.5, WeA05.6, WeA14.2, WeA14.3, WeA14.5, WeB04.4, WeB12.3, WeB13.1, WeB13.2, WeB14.2, WeB14.3, WeB14.4, WeB14.5, WeC14.1, WeC20.4

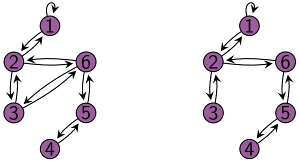
The network approach to systems is here to stay. This tutorial aims to bring to the forefront the role of graphs in these systems.

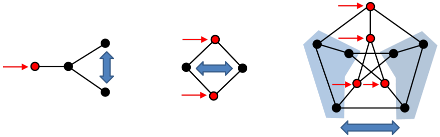
Networked Dynamic Systems



So far in this tutorial...

- graphs and modelling of network systems
- stability of network systems
- input-output properties of network systems





A Graph Structure $\Leftrightarrow$ System Behavior Morphism

We are interested in morphisms between

$$\text{(networks/operations)} \quad \Longleftrightarrow \quad \text{(systems/properties)}$$

Our thesis is that for control theoretic methods to have an impact in the growing field of networks, our techniques should be modular, scalable, and offer flexibility in their use.
Some areas that have been explored in this direction include:

- structural considerations
- compositional perspective/motifs
- approximations
- randomness

We believe this area is highly unexplored!

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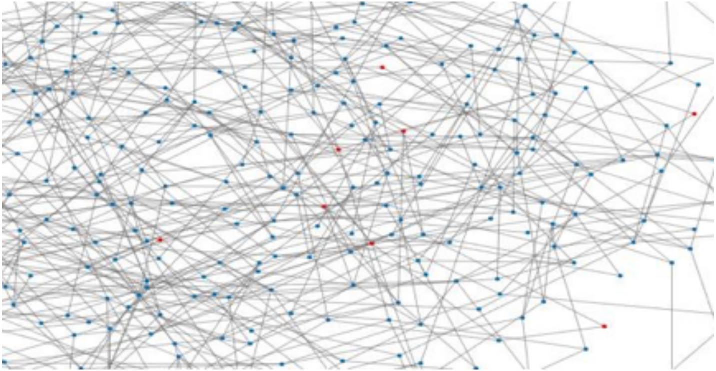
Large Scale Networks

How do we approach the analysis of networks that are too large to model?

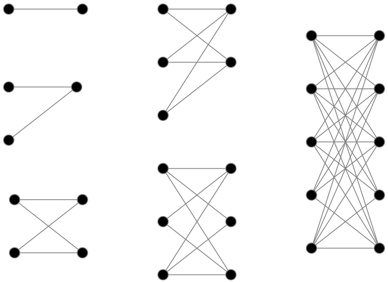


- fault detection and isolation
- power distribution networks
- transportation networks
- internet-of-things
- cyber-pysical systems
- social networks

Extremal Graph Theory

Mantel's Theorem (1907)

If a graph $\mathcal{G}$ on n vertices contains no triangles, then it contains at most $\frac{n^2}{4}$ edges.



The complete bipartite graphs are extremal

Extremal graph theory studies how global properties of a graph (i.e., number of edges) relate to local substructures (i.e., a triangle subgraph)

Forbidden Graphs

Forbidden Subgraph Problem

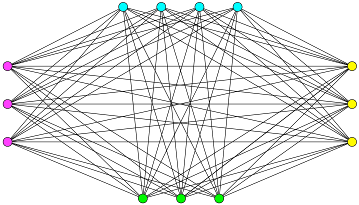
Given a set $\mathbb{H}$ of *forbidden graphs*, what is the maximum number of edges in a graph $\mathcal{G}$ on n nodes (denoted $e(\mathcal{G})$) such that $\mathcal{H} \not\subseteq \mathcal{G}$ for any $\mathcal{H} \in \mathbb{H}$?

$$\text{Extremal Number} \quad ex(n, \mathcal{G}) = \max_{\mathcal{H} \not\subseteq \mathcal{G}} e(\mathcal{G})$$

Generalize Mantel's Theorem for $\mathcal{K}_r$

Túran Graphs $T(n, r)$ - complete r-partite graphs with n vertices

$$e(n, \mathcal{K}_r) \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)$$

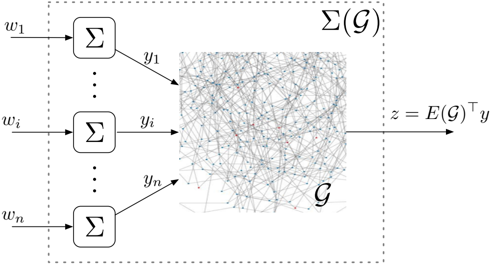


$T(13, 4)$

- avoiding paths of length k
- avoiding Hamiltonian cycles
- avoiding even length cycles
- avoiding edge disjoint cycles

Extremal Networked Systems

A simple example...



A relative sensing network

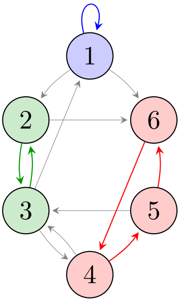
$$\|\Sigma(\mathcal{G})\|_2^2 = 2|\mathcal{E}|\|\Sigma\|_2^2$$

Proposition

Let $\Sigma(\mathcal{G})$ be a relative sensing network with n agents such that $\mathcal{G}$ is K_{r+1}-free. Then the $\mathcal{H}_2$ performance of $\Sigma(\mathcal{G})$ is at most $n^2\frac{r-1}{r}\|\Sigma\|_2^2$.

recall: k-decompositions

- k-cycle in $\mathcal{G}$: a sequence of k distinct nodes connected by edges.
- Two cycles are disjoint if they have no nodes in common.
- k-decomposition in $\mathcal{G}$: union of *disjoint cycles* covering k nodes.
  A k-decomposition is given by cycles $S_1, \ldots, S_l$ if the S_i are disjoint and $|S_1| + \cdots + |S_l| = k$.
- Hamiltonian cycle (resp. decomposition): n-cycle (resp. decomposition).

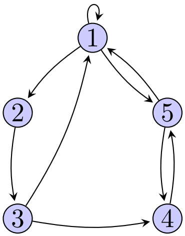


1-cycle $= (1)$
2-cycle: (23)
3-cycle: (456)
3-decomp.: $(1)(23)$ or (456)
4-decomp.: $(1)(456)$
5-decomp.: $(23)(456)$

A necessary condition for stability

Theorem[1]

A digraph $\mathcal{G}$ is stable only if it contains a k-decomposition for each $k = 1, 2, \ldots, n$



$$\begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & * & 0 \end{bmatrix}$$

An extremal question

What is the maximum number of edges in a graph $\mathcal{G}$ on n nodes before a k-decomposition appears?

[1] B. "Sparse Stable Systems", Systems and Control Letters, 2013

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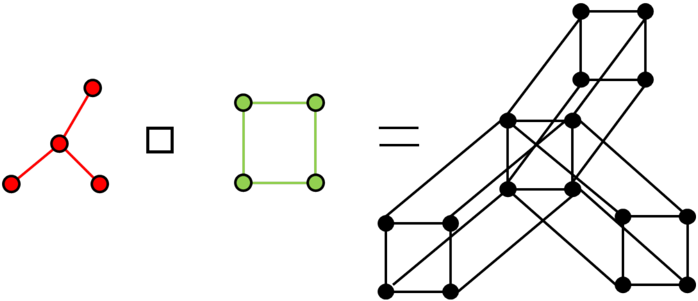
Compositional approaches: A general setup

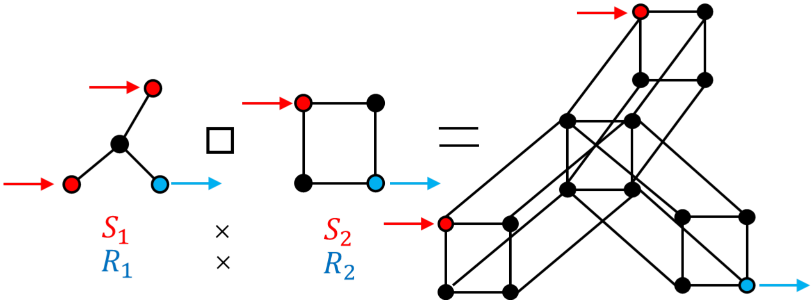
- let $\mathcal{P}$ be a system theoretic property, $\mathbf{G}$ be a class of graphs, and consider $\mathcal{P}(\mathbf{G})$
- consider a subset of $\mathbf{G}$ and examine how $\mathcal{P}$ varies over this subset
- impose algebraic operations on $\mathbf{G}$ and examine how $\mathcal{P}$ behaves with respect to this algebra
- make $\mathbf{G}$ a semi-lattice and examine how the ordering on $\mathbf{G}$ is reflected on $\mathcal{P}$

Case in point: Composite networks

Controllability of the product networks?



Input and Output Set Product



Controllability Factorization - Product Control

Theorem 1: Product Controllability

The dynamics

$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

where $A(\prod_{\square}\mathcal{G}_i)$ has simple eigenvalues is controllable/observable if and only if

$$\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$$
$$y_i(t) = C(R_i)x_i(t)$$

is controllable/observable for all i.

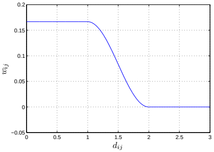
Network Learning

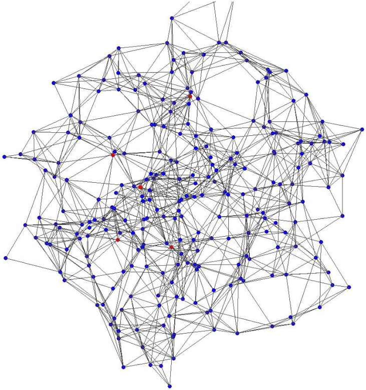
- Sensing accuracy/ confidence is coupled to an edge state, i.e., $w_{ij}(x) = g(\|x_i - x_j\|)$
- Online performance with respect to edge state control

edge states: $x_i(t)$;
coordinated state $y_i(t)$

$$\dot{y}_i(t) = \sum_{j \in N(i)} w_{ij}(x)(y_j(t) - y_i(t))$$

$$\dot{x}_i(t) = f(x_i)$$





Questions: time-scale analysis, learning, gradient flow on space of graphs

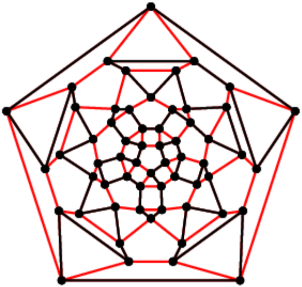
Conclusions

Graph Theory

- Algebraic graph theory
- Geometric graph theory
- Extremal graph theory
- Probabilistic graph theory
- Topological graph theory

Systems Theory

- Stability
- Performance
- Input-Output Properties
- Control Synthesis
- Control Architectures



Thank you!